CASE STUDY

Deeper Discussions: Discussion Based Tasks for Developing Problem-Solving and Critical-Thinking Skills

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Abstract

Problem-solving and, more generally, critical thinking are key components of being a mathematician and some of the most desirable skills for employers of STEM graduates. This article presents a novel learning activity, called Deeper Discussions, for developing such skills within mathematics education. The strategy revolves around discussion activities where students tackle problems that are often ill-formed or open. By leveraging pedagogic theories such as situated learning, dialogic teaching and problem-based learning, classroom dialogue is used as a mechanism for modelling these cognitive skills and enabling students to practise those skills. In this article, we present a detailed outline of these activities, the motivation behind them and their pedagogical underpinnings. We conclude the article by discussing considerations for this method moving forward.

Keywords: Problem-solving, critical thinking, problem-based learning, situated-learning, sociocultural learning, dialogic teaching, Socratic questioning, interleaving.

1. Introduction

Problem-solving is an integral part of being a mathematician. In fact, Halmos claimed that,

"*The mathematician's main reason for existence is to solve problems, and that, therefore, what mathematics really consists of is problems and solutions*" (Halmos, 1980).

Moreover, problem-solving is one of the most sought-after skills from STEM graduates (Holik, 2023, McGunagle and Zizka, 2020). Hence, the development of problem-solving and, more generally, critical thinking skills needs to be put at the heart of mathematics education. The traditional lecture would seem not to be the best for achieving this (Klegeris et al., 2017). However, it can be difficult to move away from this mode of delivery given the linear nature of mathematics, where subsequent ideas need previous concepts to be firmly grasped. This raises a question of how we can embed critical thinking into a standard mathematics unit in a way that is pedagogically sound and that genuinely represents how mathematicians approach their work.

In this article, we present a learning activity that addresses this problem by putting the development of higher-order thinking skills at the heart of learning. The activity, called a *Deeper Discussion* (DD), involves asking students to tackle broad and often ill-defined problems, primarily in tutorials in small groups. Then, using well-established pedagogies such as dialogic teaching, situated learning and problem-based learning, classroom discourse is leveraged to model problem-solving skills and to give students opportunities to engage with those skills. Moreover, with the answers to these problems providing a basis for concepts introduced later in the course, DDs form an integral part of the learning in the courses where they are used.

We start in section 2 with a general description of the strategy, followed by an explanation of the pedagogical frameworks that underpin the method. We then proceed to give a detailed explanation of how the tutorial discussions for these DDs are delivered. Lastly, we have a very brief discussion about future plans for DDs.

2. What are DDs?

2.1. A general description

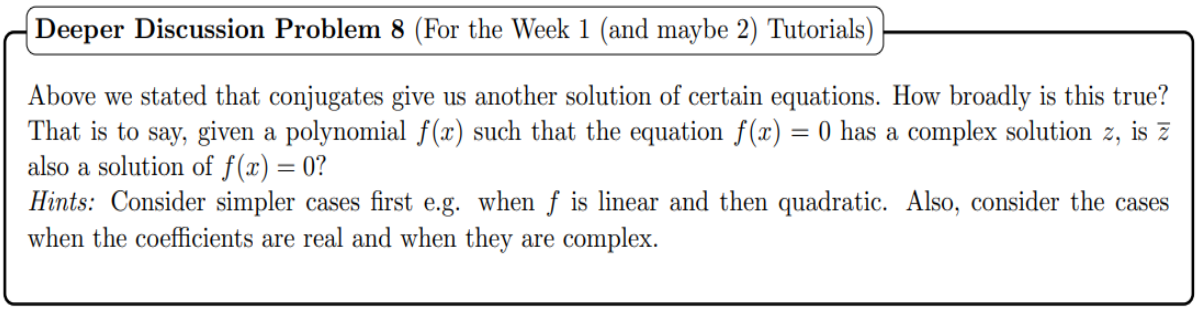
The general strategy for the DDs is as follows:

1. The problems are shared with students before tutorials in either lecture notes or exercise sheets for students to consider in their own time.
2. The problem is then discussed in tutorials. This discussion has the following three phases
   - **Phase 1:** A class discussion of the problem, where the problem is recapped, set in context and possibly analysed.
   - **Phase 2:** Students attack the problem in a small group of about two or three students.
   - **Phase 3:** A class discussion where ideas are brought together to form response to the problem. This response may be an answer to the problem or a reformulation of the problem. In the case of the latter, the class returns to phase 2 and, hence, this process is often cyclical.
3. The answer attained is often revisited in later lectures as a basis for learning further content.

Note that in what follows, we will refer to the person leading the tutorial as the "tutor".

2.2. Theoretical Bases of the Approach

Since the main purpose of DDs is to develop certain skills, an apprenticeship approach has been adopted for these tasks. In particular, the construction and delivery of DDs aligns with Lave and Wenger's (1991) *Situated Learning* (SL) model of learning, which views learners as apprentice members of what they call *communities of practice*, who are attempting to become full members. DDs have most of the core qualities of SL. For instance, DD tasks offer *authentic practice*, which is key to SL. Authentic practice is the use of learning tasks that reflect the practise of the professionals in the given community of practice. As such, the tasks elicit authentic skills. To reflect this, DD problems are intended to authentically represent the *kind* of problem that mathematicians consider. Figure 1 below presents an example of such a problem.



Deeper Discussion Problem 8 (For the Week 1 (and maybe 2) Tutorials)

Above we stated that conjugates give us another solution of certain equations. How broadly is this true? That is to say, given a polynomial $f(x)$ such that the equation $f(x) = 0$ has a complex solution z, is $\overline{z}$ also a solution of $f(x) = 0$?

Hints: Consider simpler cases first e.g. when f is linear and then quadratic. Also, consider the cases when the coefficients are real and when they are complex.

Figure 1. An example of a Deeper Discussion problem on a Foundation Year mathematics unit, MATH19812 at the University of Manchester.

While almost all mathematicians would consider the problem in Figure 1 as "answered" now, it draws upon themes, such as the solvability of equations and the structure of number systems, that are of profound interest to mathematicians and have been so for millennia.

Moreover, this problem authentically represents mathematical inquiry in style as well as in content. Mathematicians generally ask overarching and ill-defined questions which they then reduce to exact formal ones. This is reflected in DD 8 above (figure 1) where the notion of "polynomial" is ill-defined. More precisely, the coefficients are not specified. The problem requires the students to identify this

ambiguity and then critically consider possible options. Notice that there are various legitimate choices here, and the answer changes with different choices for these coefficients. Thus, these problems call upon students to use critical thinking skills which are generally not required in traditional mathematics problems which tend to be closed and exact.

Further to the authenticity of these problems, their answers serve as motivation and a theoretical foundation for latter content. This reflects the fact that in mathematics, new concepts arise out of tackling (general) problems such as these. This also fits into the *Problem-Based Learning* framework, which has been shown to improve problem-solving and critical thinking skills in mathematics (Aba-Oli et al., 2025). On a course level, this is also using the *interleaving* technique, which can also help students apprehend mathematical concepts (Rohrer et al., 2015).

Learning as Social Participation is another core component of SL. This is the idea that knowledge is co-constructed through social interaction within the given community of practice when engaged in an authentic task. This is based on Vygotsky's (1978) idea of *sociocultural learning*. Learning in DDs is almost entirely dependent on social interaction in the form of discourse. The choice of using dialogue in this way is heavily influenced by the ideas of Vygotsky (1962) and, especially, Sfard (2008), who conceptualise certain kinds of thought as a form of communication. Under this framework, the problem-solving process can be viewed as a form of dialogue, be it internal or external. As such, classroom discourse becomes a modality for modelling and practising such thinking processes.

To practically achieve this in the classroom, DD delivery leans heavily into *Dialogic Teaching* strategies (Alexander, 2020). This is a pedagogical approach that emphasises the use of classroom dialogue to "engage interest, stimulate thinking, advance understanding, expand ideas, and build and evaluate arguments, empowering students for lifelong learning and democratic engagement" (Alexander, 2025). In particular, the method of *Socratic Questioning* is extensively used in the delivery of DDs. This is a questioning strategy, dating back to Socrates, where an idea is interrogated by a series of questions that reduce it to the fundamental principles it is built on (Paul and Elder, 2007). Clearly, this lends itself very well to a logically deductive practise such as mathematics. It allows the tutor to map their own thinking process towards a solution with a series of questions. Moreover, it offers a powerful tool for incorporating students into this dialogue. Using it this way, the tutor does not simply take students' answers but, rather, probes them with further questioning, e.g. asking why they think that. This not only clarifies the students' understanding to the tutor but also trains the student to interrogate and evaluate their own thoughts with the questions the tutor poses.

Using Bloom's (Anderson and Bloom, 2014, Bloom, 1956) taxonomy can also prove to be very useful for planning and executing questioning strategies. By ascertaining the students' current understanding, the taxonomy allows us to determine the types of questions that may be most effective for helping students make progress with the problem. Such questioning not only highlights refinements to their ideas, but also to the students' own questioning process. Hence, such a questioning strategy is also a feedback mechanism, pointing out to students how they might develop their critical thinking skills.

DDs also touch upon *Legitimate Peripheral Participation*, which is another facet of SL. This is the idea that new members of a community of practice start at the periphery of the community and gradually move further inwards through authentic participation. As mentioned earlier, DD problems are intended to represent questions that are of genuine and deep interest to the mathematical community. By taking on authentic tasks, students are required to act as novice mathematicians, and hence, it is hoped that students will come to believe that they are mathematicians. Moreover, the authenticity of the problems conveys cultural perspectives of the mathematical community, something that can be consolidated by the discussions in tutorials with the tutor relaying *why* this

problem is important to mathematics. This allows students to adopt these cultural perspectives, should they wish to.

2.3. Delivery of DD discussions in Tutorials

The desired critical thinking skills are primarily developed in the tutorials for these DDs. As such, this section expands on the delivery and structure of the tutorials. In particular, we give more details on how the pedagogic strategies mentioned in the previous section are implemented.

2.3.1. Phase 1: Pre-Discussion

The main purpose of this phase is to recap the problem and set this problem in the context of the course. This is important for underscoring the authenticity of the problem. Each DD problem arises naturally while studying related ideas on the course. For example, DD 8 in Figure 1 appears when first encountering complex conjugates. Conjugates are introduced after observing that solutions of certain polynomial equations are conjugate pairs. DD 8 is then posed to explore to what extent is this true. While this would be stated in lectures, the tutor is asked to reinforce this in phase 1 of the tutorial.

While recapping the problem it is natural to review key terms e.g. complex conjugates. However, the ill-defined nature of DDs makes this a richer process and offers choices regarding to what extent the tutor unpacks the ill-defined terms at this stage. For instance, with DD 8, the tutor can choose to define that a polynomial as an expression of the form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, where $a_n, \ldots, a_0$ are simply stated to be *numbers*. This connects the students to their prior knowledge of the word "polynomial", but at the same time, it gives them an opportunity to practise critical thinking in phase 2 by identifying the ambiguity in the word "number" and then considering how it can be interpreted here. The ambiguity can then be brought to the fore in phase 3 (see section 2.2.3. below).

While this phase is led by the tutor from the front, student participation is integral for not only engagement but also to enjoin students in the critical thinking process. The tutor is asked to do this using the questioning strategies mentioned in the previous section.

2.3.2. Phase 2: Small Group Discussions

This is the main opportunity for the students to practise the desired cognitive skills. The questioning in the first phase would allow those asked to practise the desired thinking skills in an impactful way. However, this phase offers a much more accessible and direct opportunity for all students to participate in the discourse, and therefore to practise those skills.

As with all apprenticeship models of learning, it is important for the tutor to still provide guidance at this stage. For this reason, the tutor must circulate and talk with the groups. However, in these scenarios the role of the tutor is not to provide answers. Instead, it is to elucidate the students' understanding and how it can be developed using questions informed by the Socratic method and Bloom's taxonomy. For example, the tutor may ask,

- *Why do you think that?*
- *In what context is that true?*
- *What if we change X to Y?*

This also provides very direct feedback, offering a refinement of the questioning used in that specific group.

2.3.3. Phase 3: Synthesising and Evaluating Ideas

In this phase, ideas that students have developed in phase 2 are brought together in a class discussion with a view to providing a response to the question. This response might be a direct answer, but it can also be a reformulation of the question or a mixture of the two. The *cumulative* feature of dialogic teaching (Alexander, 2020) is very prominent in this phase and is important for accessing higher-order thinking skills such as analysing, evaluation and creativity (Anderson and Bloom, 2014). The tutor not only models these skills here but also makes the students participants of the thought process by carefully piecing together different ideas shared by students in phase 2 to form response to the problem. For example, when tackling DD 8 above, it is likely that most groups will assume the coefficients of the polynomials are real numbers, but some may notice that the question allows for complex coefficients. If so, the tutor can ask one of each type of group to share how they started to address the problem in this phase. In doing so, they can point out that the problem breaks down into two sub-problems, one where the coefficients are restricted to the real numbers and the other when the coefficients are permitted to be complex. Moreover, if the tutor had identified groups that made progress in any of these sub-cases, they can ask such groups what they attempted at this stage. Alternatively, the tutor can now return to phase 2 and ask the students to consider one or both sub-cases in their groups without attempting to solve them now. This would enable the entire class to consider these more precise questions, which many may not have done yet if they had not derived the questions previously.

By using students' ideas to build the response, the students are given greater ownership of the development of concepts on the course, which in turn creates a form of legitimate peripheral participation.

3. Discussion: Going forward

Having developed the strategy, it would seem appropriate to now measure the impact of DD tasks. While feedback from students has been obtained, this was primarily for professional and course development purposes. Hence, the immediate next steps would be to obtain research-standard student feedback on their perception of the impact of these activities on their critical-thinking skills and understanding of concepts.

To follow on from this, it would also be valuable to obtain a more objective measure of the impact that the strategy has on students' problem-solving and critical thinking skills. This could potentially be done using pre- and post-assessments of their critical thinking skills. Moreover, the MicroDYN approach developed by Greiff et al. (2012), may offer a way to develop an assessment instrument for these purposes. However, a thorough study of this method would need to be done to ascertain whether this is the case and, if so, how.

4. References

Aba-Oli, Z., Koyas, K. & Husen, A. 2025. Higher-order thinking skills-oriented problem-based learning interventions in mathematics: A systematic literature review. *School Science and Mathematics,* 125**,** 214-231.

Alexander, R. J. 2020. *A dialogic teaching companion / Robin Alexander,* Abingdon, Oxon ;, New York, NY.

Alexander, R.J. 2025. *Dialogic Teaching* [Online]. Available at: https://robinalexander.org.uk/dialogic-teaching/ [Accessed: 12/01/2026 2025].

Anderson, L.W. and Bloom, B.S. 2014. *A taxonomy for learning, teaching, and assessing : a revision of Bloom's [Taxonomy of educational objectives] / Lorin W. Anderson [and others],* Harlow, Pearson.

Bloom, B.S. 1956. *Taxonomy of educational objectives : the classification of educational goals. Handbook 1, Cognitive domain / by a Committee of College and University Examiners ; Benjamin S. Bloom, editor ... [et al.],* London, Longman Group Ltd.

Greiff, S., Wüstenberg, S. and Funke, J. 2012. Dynamic Problem Solving:A New Assessment Perspective. *Applied Psychological Measurement,* 36**,** 189-213.

Halmos, P.R. 1980. The Heart of Mathematics. *The American Mathematical Monthly,* 87**,** 519-524.

Holik, I. and Sanda I.D. 2023. The Possibilities of Developing STEM Skills in Higher Education. . *Athens Journal of Technology & Engineering.,* 10(1)**,** 9-26.

Klegeris, A., McKeown, S.B., Hurren, H., Spielman, L.J., Stuart, M. and Bahniwal, M. 2017. Dynamics of undergraduate student generic problem-solving skills captured by a campus-wide study. *Higher Education,* 74**,** 877-896.

Lave, J. and Wenger, E. 1991. *Situated learning : legitimate peripheral participation / Jean Lave, Etienne Wenger,* Cambridge, Cambridge University Press.

McGunagle, D. and Zizka, L. 2020. Employability skills for 21st-century STEM students: the employers' perspective. *Higher education, skills and work-based learning,* 10**,** 591-606.

Paul, R. and Elder, L. 2007. Critical Thinking: The Art of Socratic Questioning. *Journal of developmental education,* 31**,** 36-37.

Rohrer, D., Dedrick, R.F. and Stershic, S. 2015. Interleaved Practice Improves Mathematics Learning. *Journal of educational psychology,* 107**,** 900-908.

Sfard, A. 2008. *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing,* Cambridge, Cambridge University Press.

Vygotsky, L. and Cole, M. 1978. *Mind in society : the development of higher psychological processes / L.S. Vygotsky ; edited by Michael Cole [and 3 others],* Cambridge, MA, Harvard University Press.

Vygotsky, L., Hanfmann, E. and Vakar, G. 1962. *Thought and language / Lev Vygotsky, Eugenia Hanfmann, Gertrude Vakar,* Cambridge, Massachusetts, MIT Press.