CASE STUDY

Playing Games: A Case Study in Active Learning Applied to Game Theory

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Abstract

This paper describes two active learning activities which aim to introduce students to the game theoretic concepts of best response dynamics and repeated game analysis. An overview of some literature on active learning and the benefits therein is provided. This highlights that activities such as the one described in this manuscript, not only help engage students but more importantly improve their learning and understanding. The final section of this work describes how these activities fit in the pedagogic framework of a particular undergraduate mathematics class. Students generate data that can be used as context for the understanding of theoretic concepts. It is suggested that this framework is not restricted to the subject of game theory.

Keywords: Active Learning, Game Theory, Prisoner's Dilemma.

1. Introduction

Modern pedagogic theories as to how learning takes place such as constructivism and socialism. Illeris (2009) and Jordan et al*.,* (2008), indicate that an **active learning** approach is of benefit to student learning. As stated in Prince (2004) there are a variety of complementary definitions of active learning, however the general definition given in Prince (2004) is the one assumed in this paper:

"Active learning is generally defined as any instructional method that engages students in the learning process. In short active learning requires students to do meaningful learning activities and think about what they are doing."

One could argue that all learning is active as simply listening to a lecture is perhaps taking part in a 'meaningful learning activity', however as stated in Bonwell and Eison (1991) active learning is understood to imply that students:

- Read, write, discuss, or engage in solving problems;
- Engage in higher order tasks such as analysis, synthesis and evaluation.

A variety of studies have highlighted the effectiveness of active learning (Hake, 1998; Prince, 2004; Freeman et al*.,* 2014). These two papers are in fact meta studies evaluating the effectiveness of an active student centred approach. Note that the definition used in Freeman et al*.,* (2014) corresponds to simply any pedagogic approach in which students are not passive consumers of a lecture during the class meeting. Some examples of active learning in a variety of subjects include:

- The flipped learning environment in a Physics class: Bates and Galloway (2012);
- Inquiry based learning for the instruction of differential equations: Kwon et al*.,* (2005);
- Using collaborative learning in a pharmacology class: Depaz (2008).

The above sources (and references therein) generally discuss the pedagogic approach from a macroscopic point of view with regards to the course considered. This manuscript will give a

detailed description of two particular active learning activities used in the instruction of Game Theoretic concepts:

- Section 2.1 will describe an in class activity used to introduce students to the topic of best response dynamics and dominated strategies (Macshler et al., 2013);
- Section 2.2 will describe an implementation of Axelrod's tournament (Axelrod 1980a; 1980b).

These activities aim to introduce participants to concepts and aspire to their curiosity as to the underlying mathematics. Note that if there is any doubt as to the effectiveness of active learning approaches, for example Andrews et al*.,* (2011), which identifies no such relationship, inciting curiosity and engagement are still beneficial to the students' learning. Indeed in Poropat (2014) the greatest predictors of academic performance are identified not as general intelligence (Wright, 1905), but personality factors such as conscientiousness and openness.

Section 2 will describe the activities and Section 3 will detail how these fit in a more general pedagogic context. Finally, all source files for this paper (including data and the analysis) can be found at the url: https://github.com/drvinceknight/Playing-games-a-case-study-in-active-learning.

2. An exemplar: a course in game theory

Game Theory as a topic is well suited to approaches that use activities involving participants as players to introduce the concepts, rules and strategies for particular games and/or theorems presented.

In Brokaw and Merz (2004), one such activity is presented: a game that allows players to grasp the concept of common knowledge of rationality. Another good example is: Yale's Professor Polak's course (Polak, 2008), the videos available at that reference (a YouTube playlist) all show that students are introduced to every concept through activity before discussing theory (this is akin to the framework discussed in Section 3.

Just as the activity presented in Brokaw and Merz (2004), the activities presented here are suited for an early introduction to the concepts (although the activity of Section 2.2 is potentially better suited to being used at a later stage). Furthermore, these activities have also been used successfully as outreach activities for high school students with no knowledge of further mathematics.

2.1. Best response dynamics and dominated strategies

The first step in this activity and potentially before any prior description of Game Theory is to invite participants to answer the following simple question:

What is a game?

Through discussion the participants will usually arrive at the following consensus:

- A game must have a certain number $N \geq 1$ of players;
- Each player must have available to them a certain set of strategies that define what they can do;
- Once all players have chosen their strategy, rules must specify what the outcome is.

This corresponds to the general definition of a strategic form game. The main goal of this activity is to not only understand the vocabulary but also the important concept of best response dynamics which aims to identify what is the best option given prior knowledge of all other players (Maschler et al*.,* 2013). A particular game that can be analysed using base response dynamics is often referred to:

The two thirds of the average game.

A good description of the game and the human dynamics associated to it is given in Nagel (1995). The use of this game in teaching is not at all novel (The Economics Network, 2013). The rules are as follows:

- All players choose a number between 0 and 100;
- The player whose guess is closest to $\frac{2}{3}$ of the average of the choices wins.

To make use of this game in class as an introduction to the concept of best response dynamics students are handed a sheet of paper inviting them to write down a first guess. After this initial play, a discussion is had that demonstrates that the equilibrium for this game is for all players to guess 0. This is shown diagrammatically in Figure 1.

Following this discussion students are invited to play again and write down their second guess. All of the results are collected, the author has used paper forms but an automated approach could also be used. In general the input and analysis of the data takes less than 10 minutes and can be done by a helper during another class activity. Following this, the results (corresponding to the results of Figure 2a) are shown and discussed. This discussion usually revolves around the observation that not everyone acted rationally and second that some participants felt like they should 'spoil' the game by guessing larger in the second round.

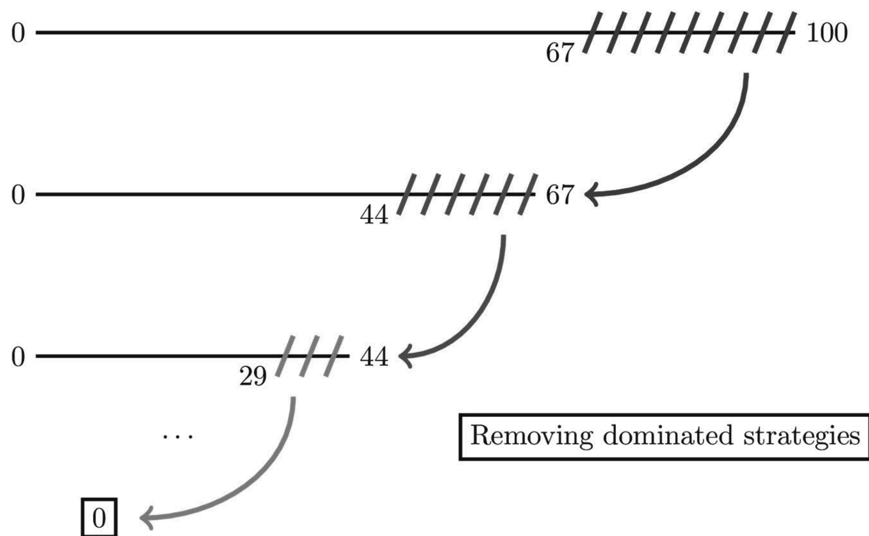
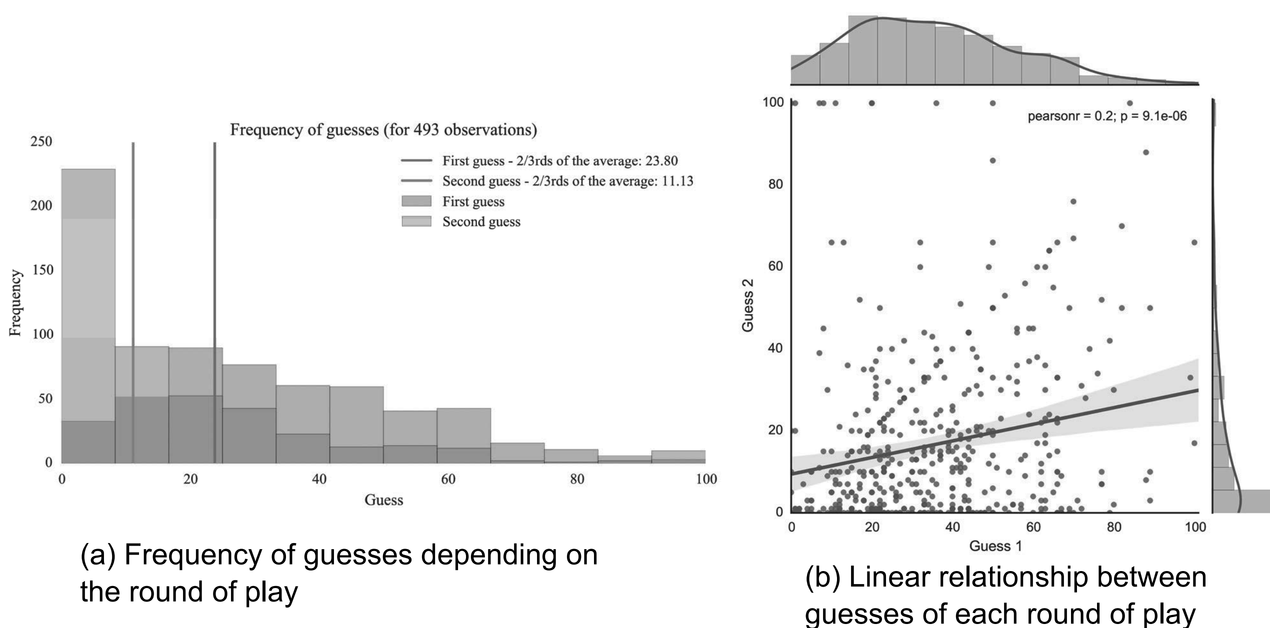


Figure 1. Equilibrium behaviour in the two thirds of the average game.

The author has used this activity on a large number of occasions and at all times collected the data. Figure 2a shows the distribution of the guesses (depending on the round of play).



(a) Frequency of guesses depending on the round of play

(b) Linear relationship between guesses of each round of play

Figure 2. Results from all data collected.

We see that the second round (after the rationalisation of play described in Figure 1) has guesses that are closer to the expected equilibrium behaviour. Figure 2b confirms this showing the linear relationship (albeit a weak one with $R^2 = 0.2$):

$$(\text{Second guess}) = 0.203 \times (\text{First guess}) + 9.45 \tag{1}$$

The fact that the coefficient of the relationship (0.203) is less than one highlights that the second guess is in general lower than the first guess. As can be seen in Figure 2(b) not all students reduce their guess. Figure 3 shows the results when removing these irrational moves. In this particular case the linear relationship is in fact stronger $R^2 = 0.43$:

$$(\text{Second guess}) = 0.33 \times (\text{First guess}) + 0.20 \tag{2}$$

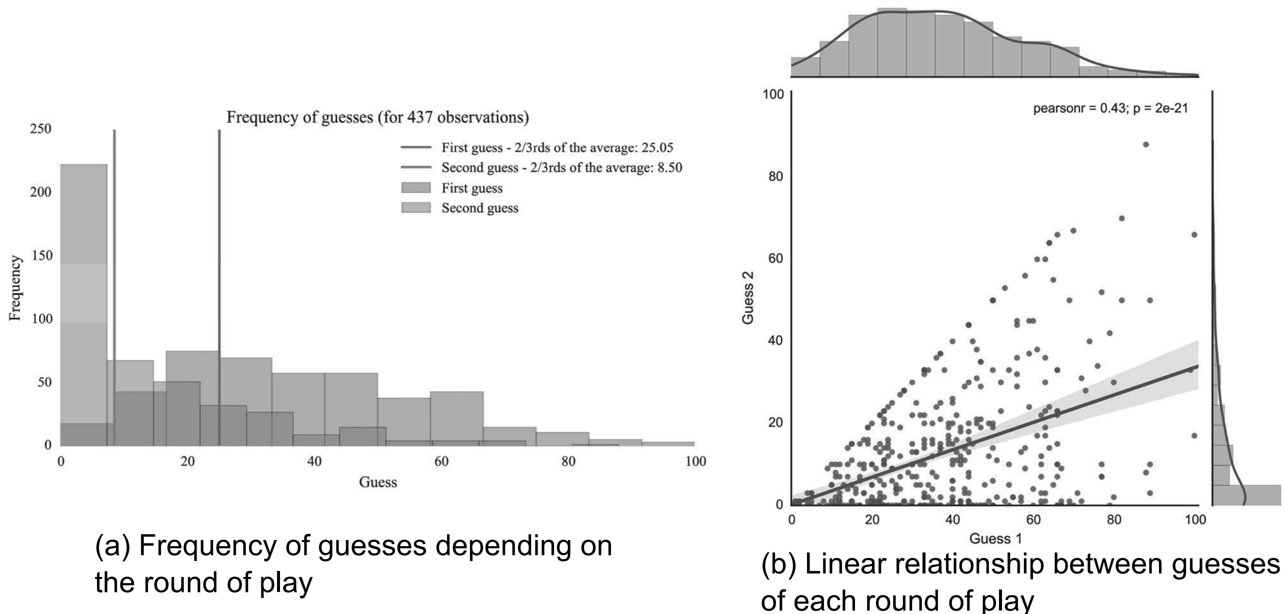
Finally, if time permits (and depending on the level of the participants), the linear relationship of (1) is used to discuss what would happen if more rounds were to be played. In particular it is possible to discuss ideas of convergence (cobweb diagrams in particular) when generalising (1) to be:

$$\text{Guess}_{n+1} = 0.203 \times \text{Guess}_n + 9.45 \tag{3}$$

To summarise this activity has the following steps:

1. Participants are explained the rules and play one round of the two thirds of the average game.
2. A rationalisation and explanation of equilibrium behaviour is described.
3. Participants play another round.
4. Results are analysed and discussed.

This activity is still quite passive in terms of physical activity (participants are seated throughout). Nevertheless it allows the data used for the discussion of the theory to come directly from the participants. Furthermore all students are active participants and there are no difficulties with regards to encouraging participation (references to these are discussed in Rocca, 2010).



(a) Frequency of guesses depending on the round of play

(b) Linear relationship between guesses of each round of play

Figure 3. Results from data when removing increasing guesses.

At the time of writing this article, software is being written to help quickly analyse plays of the tournaments (and produce the graphs seen above). Documentation for this can be found at: http://twothirds.readthedocs.org

2.2. Repeated and random games

This activity is used to introduce students to the concepts of repeated games (Maschler et al*.,* 2013). The mathematical details can be omitted from the initial description of the activity to the participants but for completeness they are included here.

A repeated game is played over discrete time periods. Each time period is indexed by $0 < t \leq T$ where T is the total number of periods. In each period N players play a static game referred to as the **stage game** independently and simultaneously selecting actions. Players make decisions in full knowledge of the **history** of the game played so far (i.e. the actions chosen by each player in each previous time period). The payoff is defined as the sum of the utilities in each stage game for every time period.

One of the most renowned repeated games is referred to as **Axelrod's tournament** (Axelrod, 1980a; Axelrod, 1980b), which is what is recreated in this activity.

Initially a description of the prisoner's dilemma is given. The prisoner's dilemma is a simple two player game that is often used to introduce the very basic notions of game theory. It is described by the following two matrices:

$$A = \begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 0 \\ 5 & 1 \end{pmatrix}$$

The row player has utility given by A and the column player has utility given by B. The strategies available to each player are to cooperate: C or to defect: D. Playing C corresponds to players choosing their first row/column and D, the second row/column.

Thus if both players cooperate they both receive a utility of 3, if one player defects, the defector gets a utility of 5 and the cooperator a utility of 0. Finally if both players defect they receive a utility of 1. As players (in this framework) aim to maximise their score, the Nash equilibrium for this game is for both players to defect.

After describing this activity and in particular explaining the simple mathematical idea of **dominated strategy** (which is what is used in the activity of Section 2.1) participants are made aware of the concept of Nash equilibrium (This in turn can lead to a brief description of the tragic yet brilliant life of John Nash).

At this point the activity is described:

1. All participants will form four groups/teams;
2. Teams will 'duel' each other in repetitions of 5 to 8 rounds (depending on available time).
3. All teams will play in a round robin tournament with cumulative scores being recorded.
4. The victorious team will be the team with the highest total score.

The tournament is run with all participants present (even those not involved in a duel). All participants are invited to stand and confer in their teams. The importance of standing (as a physical activity) is noted in Donnelly and Lambourne (2011) (whilst that reference is mainly concerned with the impact of activity on physical well-being it also describes advantages in terms of concentration). Before every round of every duel, opposing teams are encouraged to discuss strategies, after which they face away from each other and following a prompt hold up a card indicating either C or D. Duels are recorded on a wall-board in a table similar to the ones shown in Tables 1 and 2. Table 1 shows two strategies, which constantly cooperate (thus obtaining a utility of 3 in each round). Table 2 shows an example where a strategy that is alternating plays against a strategy that always defects. Figure 4 shows a photo of a final board for a particular implementation of this activity.

Table 1. Playing Tit for Tat against Cooperator

Tit for Tat	3	6	9	12	15
Cooperator	3	6	9	12	15

Table 2. Playing Alternator against Defector

Defector	5	6	11	12	17
Alternator	0	1	1	2	2

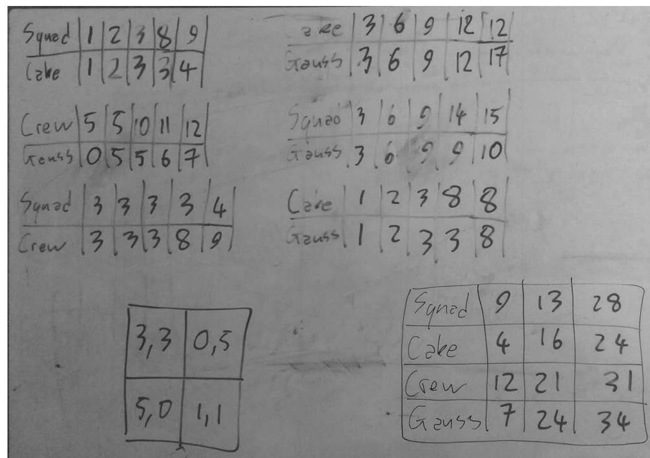


Figure 4. A photo of an actual implementation of the tournament.

The names of the strategies shown in Tables 1 and 2 are strategies that were used in the original tournaments run by Axelrod (Axelrod, 1980a; Axelrod, 1980b). The interesting fact of repeated games and one that usually becomes apparent to participants through the activity is that whilst repeating the stage Nash equilibrium (always defect) is indeed a Nash equilibrium for the repeated games, this equilibrium is not unique as reputation now has a part to play.

Note that if participants do not realise this, it is important to remind them that the goal is not to win each duel but to obtain a high score overall. Often during the tournament one team will (during the pre-round discussion) exclaim:

“We will cooperate until you defect, at which point we will defect throughout.”

Without realising it the participants have described a well-known strategy (**Grudger**) which takes in to account the entire history of play.

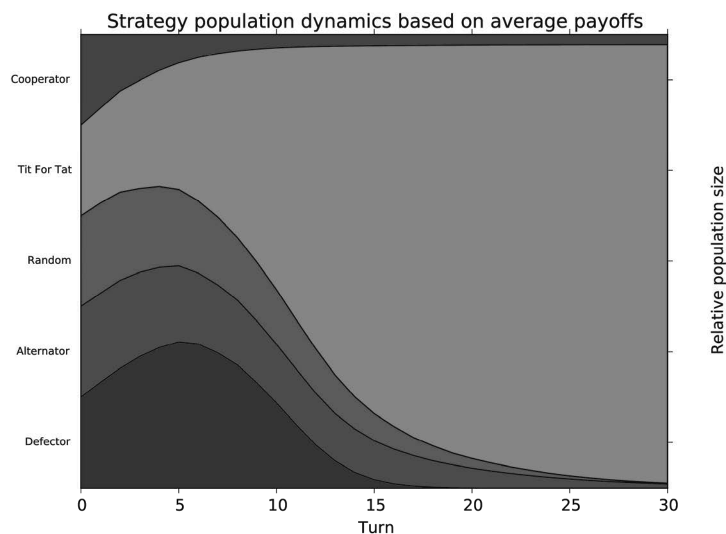


Figure 5. Repeated games in an evolutionary context.

This activity can be complemented with a demonstration of software that allows for the rapid simulation of Axelrod's tournament team (Axelrod-Python Project Team, 2015). Figure 5 shows the performance of the strategies when put in an evolutionary context.

One of the inconsistencies of this approach is that all participants observe the play by all the teams. Whilst from a mathematical perspective reputation is inferred to mean the reputation gained during a particular duel, this has the effect of teams being able to observe how other teams seem to play. A true replication of Axelrod's tournament would not allow for this. One possibility would be to invite participants to leave the room, which might be logistically constrained. From a pedagogic point of view however, having participants observe the duels often leads to a much more engaged discussion (after, as well as during the activity).

This activity is usually very enjoyable and leads to a lively discussion. Further to the fun had by participants, the theoretic discussion about repeated games can be placed in the exact context of the tournament that has just been played.

The activity can also be used to introduce further game theoretic topics with slight modifications:

- **Infinitely repeated games with discounting**: the discount factor can be interpreted as a probability of the duel continuing for another round (this can be randomly sampled);
- **Markov games**: two random game states can be a true game and an absorbing game so that this corresponds to an infinite game with discounting;
- **Evolutionary games**: this follows from considering strategies in an evolutionary context as shown in Figure 5.

3. Summary and place within a pedagogic framework

These activities have been used by the author during outreach events during which students take part in the activity of Section 2.1 and whilst the results of that are being analysed take part in the activity of Section 2.2. These two activities complement themselves and form an accessible introduction to novel mathematical topics for a wide range of age groups.

More notably however these activities have been used as part of a family of activities used in a final year undergraduate course. This particular course is taught in active learning pedagogic framework akin to a flipped class where students are introduced to theoretic concepts through prior 'playing of games'. Other examples of these activities include:

- **A rock paper scissors lizard tournament:** this serves as an introduction to mixed strategies;
- **A variety of games using coin flips:** this serves as an introduction to games with incomplete information;
- **Playing paper bin basketball in teams:** this serves as an introduction to cooperative game theory.

The general pedagogical basis for this is discussed in Section 1 and the particular framework is shown in Figure 6. Students are active participants in the creation of 'data', which drives a discussion:

- Why did you all guess this?
- Why did that team say that on that particular occasion?
- What would be a fair way of sharing the spoils for this particular game?

Following that discussion the theory can be put in context by highlighting particular theoretic results and how they correspond (or not) to the behaviour exhibited during the activity. Furthermore, this encourages immediate feedback with regards to student comprehension, which can be reactively addressed.

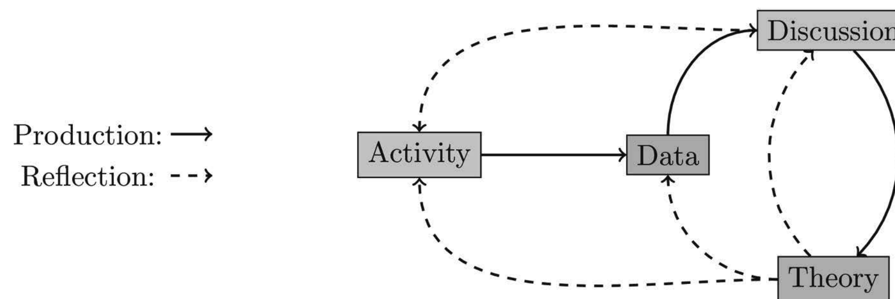


Figure 6. The active generation of data by students.

This pedagogic approach is used throughout the course (from the first lesson) and so after a few class meetings students are used to the high level of participation. Here are some examples of written feedback concerning the activities used in class:

“Classes were fun.”

“The games helped make the content interesting.”

“This course teaches me to not trust my classmates.”

Nonetheless at the start of the course certain class management techniques described in Rocca (2010) are used. For example, the extension of the ‘waiting time’ for responses to questions is implemented. For students to be active participants it is vital that they are given the time to do so.

The activities described in Section 2 are particular to game theory however the author does not feel that the general pedagogic strategy outlined in Figure 6 is constrained to a particular subject. Similar activities could be devised in other subjects where students generate ‘data’ that aids the contextualisation of theory so as to aspire to not only a constructive learning model but also a social one.

4. Acknowledgements

The author would like to thank Dr Zoë Prytherch for her help with the typesetting of this manuscript.

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