CASE STUDY

Understanding of fraction and its application in unit conversions

Aiping Xu, Mathematics Support Centre, Coventry University, Coventry, UK.
Email: aa9778@coventry.ac.uk
George Lenton, School of Computing, Electronics and Maths, Coventry University, Coventry, UK.
Email: lentong2@uni.coventry.ac.uk

Abstract

Many British university students lack confidence in manipulating fractions. In this paper, we take a detailed look into the difficulties that students are experiencing across disciplines. We also introduce a method on how to effectively convert units via manipulating fraction operations. Though this approach to unit conversions is widespread in the United States, particularly in the discipline of Chemistry, it is not well documented or applied within the UK Higher Education sector to the authors' knowledge. The method has been frequently used by the authors in Coventry's Mathematics Support Centre with very positive feedback.

Keywords: Fraction, Unit conversions, Mathematics support.

1. Introduction

In England fractions are introduced in Key Stage 1 & 2 and students are expected to grasp its four operations by Key Stage 3 (Department for Education). However, many university students still lack proficiency to some extent with fraction manipulations. Out of 40 students with grade A in A-Level Mathematics who took Coventry's diagnostic test in 1998, 17% incorrectly answered basic fraction questions such as $\frac{3}{5}\times\frac{7}{8}$ and $\frac{x}{3}+\frac{y}{2}$ (Lawson, 2003). Some mathematics undergraduates at Coventry created easily avoidable mistakes similar to those presented in Figure 1 while taking the diagnostic test.

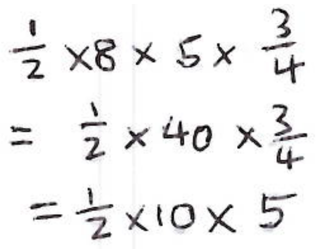


Figure 1: Example of student misconception

Though some students are able to correctly answer such questions, their approach can be inefficient, demonstrating a lack of confidence with this basic skill as well as a reliance on rote methods. Here is how a mathematics undergraduate computed $\frac{3}{5}\times\frac{7}{8}$ in the diagnostic test in 2016.

$$\frac{3}{5}\times\frac{7}{8}=\frac{24}{40}\times\frac{35}{40}=\frac{840}{1600}=\frac{84}{160}=\frac{42}{80}=\frac{21}{40}.$$

Although the working is correct, understanding fractions would have provided a much easier solution.

Many students who enter third level STEM programmes have problems with core mathematical skills (MathTEAM 2003), which includes basic fraction manipulations. Questions involving fractional indices create further difficulties, such as those presented in Figure 2.

$$= 2\left(\frac{1}{4}\right)^{-1/2} \qquad\qquad 2^{-3}$$

$$= 2\left(\frac{1}{4}\right)^{2} \qquad\qquad = \left(\frac{1}{2}\right)^{\frac{1}{3}}$$

$$= \frac{1}{(2)^3}$$

$$= \frac{1}{8}$$

Figure 2: Student calculations involving fractional indices

Sometimes engineering students incorrectly answer calculator questions such as $\frac{20\times9.81}{2.3\times1.5\times1.6}$, because although they interpret fractions as division (correct!) they input $20 \times 9.81 \div 2.3 \times 1.5 \times 1.6$ (this is incorrect), and are often unable to reason why this gives the wrong answer.

2. Typical phenomena

2.1. Addition and subtraction of mixed numbers

(a) $1\frac{1}{2}+\frac{5}{8} = \frac{\square}{2}+\frac{5}{8} = \frac{\square}{8}+\frac{5}{8} = \frac{\square}{8} = \square\frac{\square}{8}$

(b) $1\frac{1}{4}+1\frac{2}{3} = \frac{\square}{4}+\frac{\square}{3} = \frac{\square}{12}+\frac{\square}{12} = \frac{\square}{12} = \square\frac{\square}{12}$

(c) $3\frac{1}{8}-1\frac{2}{5} = \frac{\square}{8}-\frac{\square}{5} = \frac{\square}{40}-\frac{\square}{40} = \frac{\square}{40} = \square\frac{\square}{40}$

Figure 3: Examples of addition and subtraction of mixed numbers (Rayner and White, 2008)

Figure 3 above is taken from Rayner and White (2008). The idea is to guide students to first convert mixed numbers to improper fractions then add or subtract them as would be done with proper fractions. Many students have no doubts about following this method in all situations. However, splitting a mixed number into the addition of a whole number and a proper fraction creates an easy option; taking part (b) above as an example:

$$1\frac{1}{4}+1\frac{2}{3} = \left(1+\frac{1}{4}\right)+\left(1+\frac{2}{3}\right) = (1+1)+\left(\frac{1}{4}+\frac{2}{3}\right) = 2+\frac{11}{12} = 2\frac{11}{12}$$

Similar procedure can be followed for subtraction. The difference here may be considered negligable, however, such decomposition involves conceptual understanding and is far more effective when working with larger numbers such as $123\frac{1}{5}+247\frac{3}{4}$. Students too dependent on calculators or rote algorithms may fail to appreciate what is actually being done by such operations, missing opportunities for simpler solutions and developing greater understanding. Encouraging students to be explicit in their working whilst learning will help ensure that they really understand the concepts thus gives them greater confidence using such methods in the future.

2.2. Multiplication and division

Table 1 below represents some typical methods for computing multiplication and division of fractions.

Table 2: Methods for multiplication and division of fractions

Multiplication	Multiply the numerators	Multiply the denominators	Reduce if necessary
	$\frac{2}{5}\times\frac{3}{4}=\frac{6}{\ }$	$\frac{2}{5}\times\frac{3}{4}=\frac{6}{20}$	$\frac{6}{20}=\frac{3}{10}$
Division	Invert the fraction	Multiply terms	Reduce
	$\frac{2}{3}\div\frac{4}{5}=\frac{2}{3}\times\frac{5}{4}$	$\frac{2}{3}\times\frac{5}{4}=\frac{10}{12}$	$\frac{10}{12}=\frac{5}{6}$

Although these procedures will give the correct results, larger numerators and/or denominators become cumbersome to work with and are more prone to error. Whenever possible, cancellation (division) should be done first (before multiplication). For example,

$$\frac{\cancel{2}}{5}\times\frac{3}{\cancel{4}2}=\frac{3}{2\times 5}=\frac{3}{10}.$$

Such cancellation is also very effective when applied to unit conversions.

Regarding fractional division, students generally know that they need to invert the divisor fraction then multiply without comprehending why. Fractions are equivalent to divisions and students should be able to fluently interchange between them. The following might facilitate students' understanding:

$$\frac{a}{b}\div\frac{c}{d}=\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)}=\frac{\frac{a}{b}\times\frac{d}{c}}{\frac{c}{d}\times\frac{d}{c}}=\frac{\frac{a}{b}\times\frac{d}{c}}{1}=\frac{ad}{bc}.$$

Reasons:

- The first equality: division is the same as a fraction of the terms;
- The second equality: we do not want to keep a fraction as the denominator so we multiply it by its reciprocal $\frac{d}{c}$. To keep the fractions value unchanged, we should multiply the numerator by the same value.

Or we could apply the four operations of integers:

$$\frac{a}{b} \div \frac{c}{d} = (a \div b) \div (c \div d) = (a \div b) \div c \times d = \ a \div b \div c \times d = (ad) \div (bc) = \frac{ad}{bc}.$$

When being addressed in an algebraic context, this becomes more problematic if they lack conceptual understanding. A typical example is the following common mistake:

$$\frac{7x^2}{14x^2 + 21x} = \frac{7x^2}{14x^2} + \frac{7x^2}{21x} = \frac{1}{2} + \frac{x}{3}$$

The working out is wrong but can be hard for students to discover, especially when it happens as part of a larger problem. This question should be addressed as:

$$\frac{7x^2}{14x^2 + 21x} = \frac{7x^2}{7x(2x+3)} = \frac{x}{2x+3}$$

2.3. Complex numbers

The following question is taken from the engineering maths book (Singh 2003):

Express $\frac{3+j4}{j5}$ in the form of $a + jb$ where a and b are real.

(Some worked solution is provided via the online answers accompanying the Singh (2003) text at https://he.palgrave.com/resources/CW%20resources%20(by%20Author)/S/Singh/worked-solutions/Solutions_10a.pdf.)

We need to find the complex conjugate of $j5 = 0 + j5$, which is $0 - \ j5 = \ -j5$. Hence,

$$\frac{3+j4}{j5} = \frac{-j5(3+j4)}{5^2} = \frac{20 - j15}{25} = 0.8 - j0.6.$$

This is also a typical method for both mathematics and engineering undergraduates. However, if they could freely manipulate fractions, they could achieve the goal in an easier way:

$$\frac{3+j4}{j5} = \frac{1}{5} \times \frac{3+j4}{j} = \frac{1}{5} \times \frac{(3+j4) \times j}{j \times j} = -\frac{-4+j3}{5} = 0.8 - j0.6.$$

Note that we have multiplied the numerator and denominator with j. We can also use its conjugate $(-j)$, which would not make much difference.

2.4. Simplification of closed-loop transfer function

Students in Control/Mechanical Engineering at Coventry are expected to work out the closed-loop transfer function if given the open-loop transfer function. They are familiar with diagrams similar to those presented in Figure 4, and the formula $\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$.

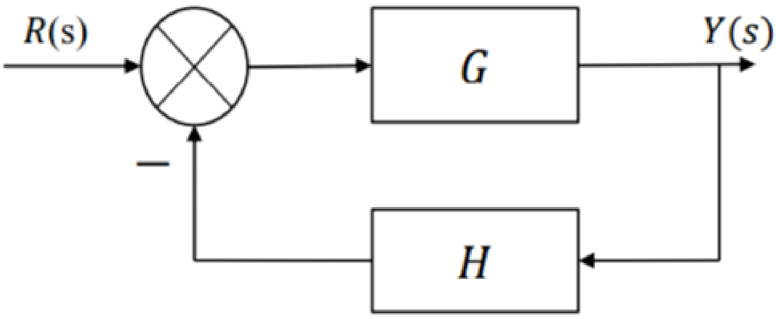


Figure 4: Closed-loop block diagram

Derivation of the formula is typically unproblematic. However, while dealing with concrete examples, many find it hard to simplify the closed-loop transfer function due to its complexity involving fractions. For example, an open-loop unstable system is to be stabilised by a feedback control with the control gain K_p in the feedback as shown in Figure 5.

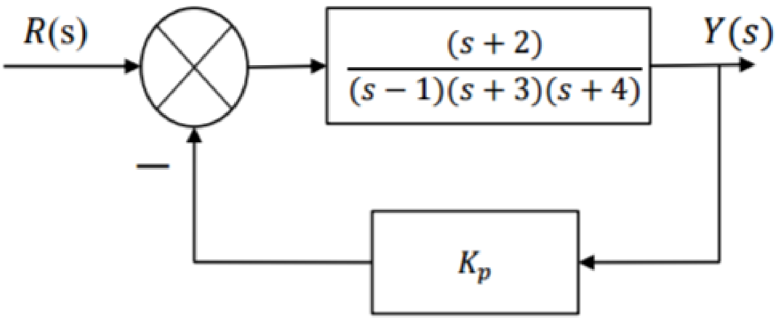


Figure 5: Diagram for closed-loop transfer function example

Obtain an expression for the overall closed-loop transfer function in its simplest form.

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

$$= \frac{\dfrac{s+2}{(s-1)(s+3)(s+4)}}{1 + \dfrac{s+2}{(s-1)(s+3)(s+4)}.K_p}$$

$$= \frac{s+2}{(s-1)(s+3)(s+4) + \ K_p(s+2)}$$

$$= \frac{(s+2)}{s^3 + 6s^2 + \ (5 + K_p)s + (2K_p - 12)}$$

Many students find the goal unachievable.

3. Unit Conversions (Dimensional Analysis)

Mistakes in related areas are everywhere. Figure 6 illustrates an example extracted from the BBC Bitesize website http://www.bbc.co.uk/education/guides/z8b9d2p/revision/10.

Q **Question**

The density of gold is 19.3 g/cm^3. Convert this into kg/m^3.

A **Hide answer**

First convert grams into kilograms.

1 kg = 1,000 g

Next convert cubic centimetres into cubic metres.

1 m^3 = 1,000,000 cm^3

Finally combine the two unit conversions.

$$19.3\ \frac{g}{cm^3} = \frac{19.3 \times 1,000}{1,000,000}\ \frac{kg}{m^3}$$

$$19.3 \times 1,000 \div 1,000,000 = 0.0193\ kg/m^3$$

Figure 6: Conversion example from BBC Bitesize website

Similar mistakes happen at British universities across the range of science disciplines.

Cancellation can be very effective when being applied to unit conversions, which play a very important role in Sciences and Engineering courses. There is a multitude of units for measuring most quantities and students are often required to convert from one to another. Students are frequently unsure whether they should multiply or divide some numbers. Dimensional Analysis, also called Factor-Label Method or the Unit Factor Method, is a problem-solving method that uses the fact that any number or expression can be multiplied by one without changing its value (Texas A&M University, n.d.; Rice University, n.d.). It is a useful technique and may be widespread in the United States, particularly in the discipline of Chemistry. The idea is to interpret numbers with units, e.g. 16.2 meters or $32ft/sec^2$, exactly the same as coefficients with variables. Consider a simple example of changing from m to cm, we need to decide which ratio equal to '1' we should multiply m by to get cm. If we use the fraction with cm on top and m on bottom, m will 'cancel out' when we simplify as follows

$$4m = 4\cancel{m} \times \frac{100cm}{1\cancel{m}} = 400cm$$

When converting units student should bear in mind the following: anything can be multiplied by ‘1’ — a carefully chosen form of ‘1’ such as $\frac{100cm}{1m}$, without altering its value. This can be applied to all unit-conversion questions, regardless of the units’ complexity. We will discuss this further with examples from different backgrounds. However, the authors have not seen anything like this here in the UK.

3.1. Nursing numeracy example

All universities and some nursing jobs require students to pass a nursing numeracy test in order to gain a place on a nursing course, or to be offered a job. This numeracy test normally consists of some drug calculation questions, which could be treated as unit conversion.

Question: How much penicillin mixture should be given for a dose of $500mg$ if the mixture comes as $250mg$ per $5ml$? (sigma Coventry University, n.d.)

Students are advised to use the formula given in Figure 5:

$$\frac{\text{What you want}}{\text{what you have}} \times \frac{\text{the stock level}}{1}$$

Figure 5: Drug calculation formula

Solution:

- What you want is $500mg$
- What you have is $250mg$
- The units are the same
- Stock volume is $5ml$
- Volume to be given = $\frac{500}{250} \times 5 = 10ml$.

Such method is frequently recommended at universities across the UK (University of Leeds, n.d.). However, many nursing students find it hard to remember formulae and they can get confused when required to place numbers in the correct places. If we apply the special ‘1’ method (Straight A Nursing, 2017), the approach here becomes: $5ml = 250mg$, we can use $'1' = \frac{5ml}{250mg}$ because we need to cancel out ‘mg’, consequently

$$500mg = \cancel{500mg}2 \times \frac{5ml}{\cancel{250mg}} = 10ml.$$

3.2. Bioscience example

Each year at Coventry about 200 students are enrolled into Bioscience courses and they are required to do a module on Quantitative Skills, where many questions are based around different unit conversions. Students can use the same technique and multiply by a well-chosen fraction that equals ‘1’, only they need to do it for each unit to be converted. For example:

Question: What is the molar concentration (in mM) of a 0.4% (w/v) $NaOH$ solution? (The molar mass of $NaOH$ is $40g/mol$)

Solution: To achieve our goal we need to perform a series of unit conversions: ($g/ml \to g/l \to mmol/l \to mM$)

$$0.4\% \quad = \frac{0.4g}{100ml} = \frac{0.4g}{\cancel{100ml}} \times \frac{\cancel{1000ml}10}{1l} = \frac{4g}{1l} = \frac{\cancel{4g}}{1l} \times \frac{1mol}{\cancel{40g}10}$$

$$= \frac{0.1mol}{1l} = \frac{0.1\cancel{mol}}{1l} \times \frac{1000mmol}{1\cancel{mol}} = 100mmol/l = 100mM$$

3.3. Engineering example

Last year the Mathematics Support Centre at Coventry had over 11,800 student visits with the majority (74%) coming from the faculty of Engineering, Environment and Computing. Many engineering students struggle with unit conversions. Some questions are similar to:

Rank the following values of stress in INCREASING order of magnitude

- $5\ kN/mm^2$
- $6000N/mm^2$
- $200\ kN/m^2$
- $80000N/m^2$

All the quantities are given in different units, we need to convert them into the same one, for example kN/mm^2, the others should then become

$$6000N/mm^2 = \frac{6000N}{1mm^2} = \frac{\cancel{6000N}6}{1mm^2} \times \frac{1kN}{\cancel{1000N}} = \frac{6kN}{1mm^2} = 6kN/mm^2$$

$$200kN/m^2 = \frac{200kN}{1m^2} = \frac{200kN}{\cancel{1m^2}} \times \frac{\cancel{1m^2}}{(10^3mm)^2} = \frac{200kN}{10^6mm^2} = 2 \times 10^{-4}kN/mm^2$$

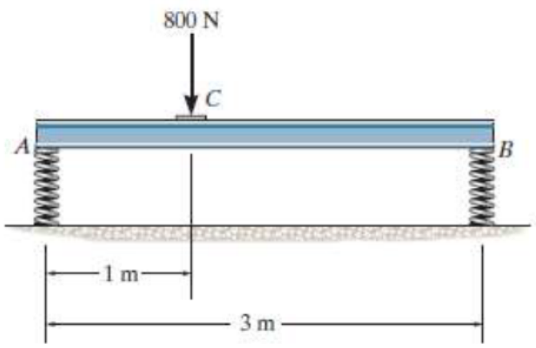
$$80000N/m^2 = \frac{80000N}{1m^2} = \frac{\cancel{80000N}80}{1m^2} \times \frac{1kN}{\cancel{1000N}} = \frac{80kN}{1m^2}$$
$$= \frac{80kN}{\cancel{1m^2}} \times \frac{\cancel{1m^2}}{(10^3mm)^2} = \frac{80kN}{10^6mm^2} = 8 \times 10^{-5}kN/mm^2$$

As long as we have the same unit, we can easily order the four quantities in increasing order as $80000N/m^2$, $200\ kN/m^2$, $5\ kN/mm^2$, $6000N/mm^2$.

4. Accuracy

Another advantage of helping students become more comfortable using fractions is to improve the accuracy of their calculations. At Coventry, around 300 students per year join the Mechanical Engineering course, the following question (Figure 6) comes from their first-year module:

The horizontal beam is supported by springs at its ends. Each spring has a stiffness of k=5 kN/m and is originally unstretched so that the beam is in the horizontal position. Determine the angle of tilt of the beam if a load of 800 N is applied at point *C* as shown.



Answers
a) 1.02^{o}

Figure 6: Mechanical Engineering example from Coventry University

Solution: We label the spring forces F_A and F_B at points at A and B respectively. To work out F_A, we take B as pivot so we have the moment equation

$$\mathrm{F_A} \times 3 = 800 \times (3 - 1)$$

Consequently,

$$\mathrm{F_A} = \frac{800 \times (3-1)}{3} = \frac{1600}{3}\ \mathrm{N}$$

Similarly we have $\mathrm{F_B} = \frac{800 \times 1}{3} = \frac{800}{3}$ N.

If we label the compressed distance X_A and X_B at points at A and B respectively, then apply Hooke's law, we have $\mathrm{F}_A = \mathrm{k\,x_A}$ and $\mathrm{F_B} = \mathrm{k\,x_B}$

$$\mathrm{x_A} = \frac{\mathrm{F_A}}{\mathrm{k}} = \frac{\frac{1600}{3}\ \mathrm{N}}{5\,\frac{\mathrm{kN}}{\mathrm{m}}} = \frac{1600}{3}\ \mathrm{N} \div \frac{5\mathrm{kN}}{1\mathrm{m}}$$

$$= \frac{1600}{3}\ \mathrm{N} \times \frac{1\mathrm{m}}{5\mathrm{kN}} = \frac{1600}{3}\ \cancel{\mathrm{N}} \times \frac{1\mathrm{m}}{5\ \cancel{\mathrm{kN}}} \times \frac{1\cancel{\mathrm{kN}}}{1000\ \mathrm{N}} = \frac{8}{75}\mathrm{m}$$

And similarly $\mathrm{x_B} = \frac{4}{75}\mathrm{m}$.

Thus the angle is $\Theta = \tan^{-1}\left(\frac{\frac{8}{75} - \frac{4}{75}}{3}\right) = \tan^{-1}\left(\frac{4}{3 \times 75}\right) = 1.02°$.

Some students do not consider fractions as numbers so they work their results with decimals. If they keep three decimal places in their working, they would have had

$$F_A = 533.333 \text{ N} \text{ and } F_B = 266.667 \text{ N}$$

$$x_A = 0.107 \text{ m and } x_B = 0.053 \text{ m}$$

As a result,

$$\tan(\theta) = \frac{0.107 - 0.053}{3} = 0.018$$

$$\theta = \tan^{-1}(0.018) = 1.031°$$

They then doubt the correctness of their working out. Greater confidence with fractions would enable students to employ them throughout the problem, increasing the accuracy of their results.

5. Conclusion

Many students still struggle with fractions at university level. They take basic operations on fraction as procedural rather than conceptual. If students could gain a deeper understanding of fractions they could answer subject-related questions much more confidently and efficiently. Confidence in handling fractions opens up further methods to the student such as the described approach to unit conversion, as well as improving their overall accuracy by utilising them throughout the problem solving process. The stated method of unit conversion has been repeatedly used by the authors in the Mathematics Support Centre at Coventry, which has been proven to be very successful. In spite of students' backgrounds, they find the method easy to understand and accept. Consequently students know what they should do so have largely boosted their confidence in applying mathematics.

6. Acknowledgements

The authors are extremely grateful to the anonymous referee for the useful comments and references on the manuscript.

7. References

Department for Education, 2013. Mathematics programmes of study: key stages 1 and 2, National curriculum in England. Available at: https://www.gov.uk/government/uploads/system/uploads/attachment_data/file/335158/PRIMARY_national_curriculum_-_Mathematics_220714.pdf [Accessed 12 May 2017].

Department for Education, 2013. Mathematics programmes of study: key stages 3, National curriculum in England. Available at: https://www.gov.uk/government/uploads/system/uploads/attachment_data/file/239058/SECONDARY_national_curriculum_-_Mathematics.pdf [Accessed 12 May 2017].

Lawson, D., 2003. Changes in student entry competencies 1991 – 2001, *Teaching mathematics and its applications*, 22(4), pp.171-175

MathsTEAM, 2003. Diagnostic Testing for Mathematics, LTSN, Birmingham

Rayner, D., and White, M., 2008. *Essential Maths 7H*, Elmwood Education

Singh, K., 2003. Engineering mathematics through applications, Basingstoke: Palgrave Macmillan

Texas A&M University, No date. Math Skills Review. Available at https://www.chem.tamu.edu/class/fyp/mathrev/mr-da.html [Accessed 1 September 2017].

Rice University, No date. Experimental Biosciences – Dimensions and Units. Available at http://www.ruf.rice.edu/~bioslabs/tools/data_analysis/dimensions_units.html [Accessed 1 August 2017].

sigma Coventry University, No date. Drug Calculations for Nurses (N12). Available at http://sigma.coventry.ac.uk/resources/nursing [Accessed 1 September 2017].

Straight A Nursing, 2017. Dosage calculations the easy way. Available at http://www.straightanursingstudent.com/dosage-calculations-the-easy-way [Accessed 1 September 2017].

University of Leeds, No date. Maths for Nurses: Unit conversions. Available at http://www.worcester.ac.uk/studyskills/documents/Nursing-unit-conversion_Leeds_University.pdf [Accessed 1 September 2017].