CASE STUDY

A calculus course in knowledge feedback format

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Abstract

This paper presents a design of an undergraduate calculus course that aims at achieving four connected goals: 1. higher student activity, 2. teacher-student exchange, 3. continuous teacher learning about the present students' mathematics knowledge as the course progresses, and 4. immediate feedback from this learning to improve the ongoing course. In this process, students can perceive the course as one where one's own activity contributes to other students' learning and even improves the course in the long term. The main content of this paper is the *course design* to fulfil or come close to these goals.

By the design it is uncomfortable for a student to be passive (breaking the course pattern), and comfortable to be active (following the course pattern). However, all course features except examination are truly voluntary. Student passivity is accepted.

Keywords: Learning, dialogue, quiz, diagnostic, course design.

1. Introduction

This course is designed for teachers who enjoy interaction with students, and are interested in listening to students' mathematical questions and to provide feedback on their level, or who wish to develop such competencies. It is based on the idea that a teacher, besides the subject knowledge itself, needs to develop a complementary knowledge: a detailed understanding of how students understand and misunderstand the subject in conjunction with practical ways to overcome such difficulties.

This complementary knowledge can be gradually and systematically developed during the service, resulting in courses developing year by year. If this learning also affects the ongoing course, students may feel included and invited to participate, which may by itself lead to an extra effort in their learning.

This course contains several orchestrated activities to provide opportunities for teachers to develop such competence, which is not possible without active students and thus must be dependent on the student group. How to design such a course is not a trivial task.

Part of the means to activate students is mini exams during some course weeks. This is popular among many students for this very reason – increased activity. But questioning, investigating and deeper learning requires a more open error-tolerant attitude than "examination-attitude". To this end, activities between the small exams are designed to have a non-examination character: investigative, open, inclusive, (slightly) mathematically provocative, and viewing all errors as constructive and as occasions of learning.

It must be stressed that resources for a systematic evaluation of the course and production of evidence that underpin the intentions that have guided course design choices have not been available (see however Section 5).

In Wittman (1995), mathematics education is described as a design science, as is engineering, architecture, business and medicine. Here the most essential question is how to design artefacts that are successful, which may be an engine, a building, an enterprise, a medical treatment method,

or a course. All these areas are strongly dependent of knowledge of different sciences, of course, but the goal of research is not only to understand. The main point is that creative design has a decisive importance in design sciences. The design must result is an artefact, that can be scrutinized. In the case of education: how a course is designed is essential for the result of it. The questions "What?" and "Why?" may be enough in basic research, but must in a design science lead to a particular answer to "How?".

A further basic inspiration for this course's design has been Donald Shön's reflective practice (Shön, 1983), in terms of professional development. Here professionals learn from their own professional experiences, and engage in a process of continuous learning.

The idea of reflective practice can be taken further when the profession is teaching. In this case there are always learners to engage in dialogue with, in the aim of reflective practice and continuous professional development. Then professional development becomes entangled with the learners' learning, and thus to the immediate goal of the professional activity.

The willingness of students to interact and share the teacher's knowledge and skills is discussed in Kindberg (2013) in a perspective that is both pedagogical and rhetorical. The teacher's ability to construct a safe and open learning atmosphere, which is also a rhetorical issue because of its focus on verbal communication, is essential for the students' learning outcome.

In the special case of mathematics education, verbal reflection is scarce by tradition, despite the subject's fundamental dialogical nature (Lennerstad, 2008, Ernest 1994). One can argue that mathematics problem solving, and mathematics in general, has a characteristic and important vagueness, making mathematics in need of verbalisation. This is expressed by P. Ernest in the preface of Rowland (2000):

"*Precision is the hallmark of mathematics and a central element in the 'ideology of mathematics'. Tim Rowland, however, comes to the startling conclusion that vagueness plays an essential role in mathematics talk. He shows that vagueness is not a disabling feature that detracts from precision in spoken mathematics, but is a subtle and versatile device which speakers deploy to make mathematical assertions with as much precision, accuracy and confidence as they judge the content and context warrant.*"

Thus, dialogue and its natural vagueness is at the core of the nature of mathematics.

2. The calculus course: the practicalities

The course 'Analys 2' is a basic calculus course at the Blekinge Institute of Technology in Sweden. Its eight weeks contains three weeks about integrals, three weeks about ordinary differential equations, and one week about Taylor series. The final week is devoted to an overview of the course's results and methods, and to old exams. The immediately preceding course 'Analys 1' covers numbers, the function concept, different function classes, limits, derivatives, and applications. Both courses are mandatory in five civil engineering programs at Blekinge Institute of Technology, and approximately 100 students from two or three of these programs participate at each course occasion. For each student and week, the course entails eight hours of lectures, and two hours of scheduled exercise solving.

The assessment of the course is in two parts: an individual written exam of 4.5 credit points and a project of 1.5 credit points. A student must pass both. In the written exam, 50% of the points are needed to pass, and no points can be gained to this exam from course activities. Each teacher has some liberty in how to design the project part – which I have used for the design of this course.

The six tasks in the written exam are conventional except for the last, which is conceptual. This task requires an account of the relations between concepts in the course, and possibly proofs of theorems. An example of such a task is the following:

"*Q6. How is differentiation and integration related? Are there elementary functions that do not have an elementary primitive function, although they are integrable? What is the difference between a definite and an indefinite integral? Which are the main methods to calculate a primitive function? Is there any differential equation that can be solved by integration? How can one prove that a continuous function on a bounded interval is integrable?*"

Grading the conceptual task has not turned out to be more difficult than other tasks, however slightly more time consuming. Also, students recognize that only solving problems, and learning proofs word by word, is not enough to pass this task.

3. Design of the course

To clarify the course design within the frames defined by Section 2; there are 2 hours lectures on Tuesday and on Wednesday, and 4 hours on Friday. The two hours of scheduled exercises occur during each week in addition to the plan described in this section.

Monday	Tuesday	Wednesday	Thursday	Friday
				ST
				Post
	ME or Post	Pre		Post
	Pre	TT		Post
DQ deadline			CQ deadline	

Weekly design DQ = diagnostic quiz, CQ = Conceptual quiz, ME = Mini exam (course week 3, 5, 7), Pre = Pre-lecture, Post = Post-lecture, TT = Teacher's tasks, ST = Students' tasks

Figure 15. Weekly plan of calculus course

3.1 The guide, the pre-lectures and the post-lectures

The content to learn during one week is presented for students by a one page document, called a 'guide' – one for each week except the last. The guide contains:

1. the new topics of the particular week;
2. what students are supposed to learn;
3. recommended exercises;
4. six 'Teacher's tasks';
5. six 'Student's tasks'.

Typically, new mathematics is 'pre-lectured' during the second half of Tuesday's and during Wednesday's lectures. 'Pre-lecturing' means giving typical examples, main ideas and connections with motivations, usually no proofs, and solving typical and not very complicated problems.

The last hour of Wednesday is devoted for the teacher to solve the Teacher's problems.

The first hour of Friday is the Students' session: students solve the Student's problems in the lecture on the whiteboard, commented on in an affirmative way by the teacher. This is not an examination, but voluntary. If no student wishes to solve a problem, the teacher does it.

The remaining hours of Friday and the first hour of Tuesday is used for 'post-lectures'. Post-lecturing means proving statements not proved in the pre-lecture, generalizing and deepening the week's mathematical content, and solving more complex problems. This can be assumed to me more fruitful after the students have calculated and attempted problems themselves.

3.2 The quizzes

To complement the problem solving activity in Student's tasks, there are two quizzes about the theory: a Diagnostic quiz and a Concept quiz. Both occur every week on the course site except during the last week. Each contains about 12 mathematical claims concerning the concepts of the week. For each quiz statement, a student must choose: True or False. After the choice, the correct answer appears immediately together with an explanation of the answer; why it is false, if it is, and under which cases it might be true, and how it connects to other concepts of the course.

These explanations provide a lot of mathematical understanding, at least about how the concepts connect with each other, and the limits of them. Here is an example of a part of a quiz.

True or false?

1. The differential equation $y' = 1/y$ is linear.

   Answer: False. Comment: Not even after multiplication into $y'y = 1$ does it have the form $f(x)y' + yg(x) = h(x)$, as it should for linear equations of the first order (for some functions $f(x)$, $g(x)$ and $h(x)$);

2. Linear differential equations can only be solved if the coefficients are constant.

   Answer: False. Comment: With the integrating factor method, many first order linear differential equations can be solved that do not have constant coefficients;

3. No differential equation is both separable and linear.

   Answer: False. Comment: The differential equation $y' = y$, for example, has both properties. In the form $y' - y = 0$ it is linear, and in the form $y'/y = 1$ the variables are separated. So it can be solved by both methods.

Three examples with "False" as right answer were here chosen, since such statements may be more difficult for us teachers to find and formulate. The Diagnostic quiz is similar, but tests the prerequisites for the next week. It is due for completion before the first lecture that week.

The Conceptual quiz is due Thursday evening, before the Students' session. They test the main ideas and connections of the week's concepts, which are presented in pre-lectures. The timing of this quiz is intended to enhance the Students' session to make general theory more present and available during the problem solving activity.

3.3 The mini exams

Mini exams occur in course week 3, 5 and 7. They concern content of week 1 and 2, 3; weeks 4 and 5; and week 6, respectively. Each mini exam is 45 minutes long and takes place the first hour of Tuesday, and includes two or three problems. Mini exams test the new concepts, but require very little calculation, due to the very short time.

3.4 Assessment of the project

For each quiz that is completed before the deadline, a student gets one point regardless of the number of correct answers. Quiz points and mini exam points are added, and this sum must be high enough to pass the project. A student who has all quiz points needs 40% of the available mini exam points to pass the project, while 80% is required with no quiz points.

3.5 The Student's sessions

On Friday mornings, students solve problems before the teacher and the entire class during lecture. They have six problems to choose between, and each student can choose either a problem that can easily be solved in class, or a harder example that may provide more learning. The Teacher's problems, solved by the teacher a few days earlier, are intended to provide inspiration. For each problem the teacher simply asks if any student want to solve it. It there is no one, the teacher solves it without disappointment, which underlines the voluntary character of the activity.

Circa 80% of the Student's problems were solved by students, by approximately 15 different students. About 60% of the circa 100 students registered were present during the lectures.

An alternative organization, that has not been tested, is to divide the room into six sections, enumerated 1-6, and then ask students to sit down at a number corresponding to the problem that one has prepared to solve. Then the teacher can ask a smaller group for a problem solver, and students are probably more focused on one particular problem.

3.5 Intentions of the course design

One intention with pre-lectures and post-lectures and problem solving in between, is that proofs are more meaningful for students after they have been active themselves. The style of a pre-lecture is indeed intended to make it possible for students to become mathematically active with the new concepts as soon as possible, and also to engage weaker students.

True-false quizzes are challenging for many students, since it is uncomfortable to provide many wrong answers, particularly if you expect the course to be easy. This is made milder (and perhaps more inviting) by the fact that the number of errors presented at the quizzes do not affect the assessment. Since the correct-incorrect feedback is unavoidable, the quizzes provide learning about particular mathematical difficulties.

This means that the choice of quiz questions is important – they must really formulate and test basic ideas. It was challenging as a teacher to produce false statements about mathematics. This is defendable since students immediately obtain a complete explanation about what is actually true, and why. An example of a false question is "*A tangent is always a polynomial of first degree*", which of course is false since some tangents are constants – polynomials of zeroth degree. Such false statements provoke students before answering to consider "*Is there some special case I am missing now?*". This is a highly appropriate scientific question, and it is desirable if it becomes habit.

The purpose of the quizzes is to make students more familiar with the theory and show how theory improves problem solving techniques. The Teacher's tasks are intended to pave the way for students to solve their Student's tasks during the Student's session a few days later. The Student's session is obviously assessment-free, characterized by the basic idea 'It is better to do the mistakes now than during examination'. Hence, the Student's sessions and the quizzes attempt to provide an atmosphere of openness about mathematical issues, deeper discussion and mathematics learning, which can be seen as a way of communicating that is opposite to that of the examination. This points towards a central issue to succeed with a course of this type: the quality of teacher-student dialogue and interaction (see next section.)

All students quickly learn that during course week 3, 5 and 7, (mini) examinations take place on the same topics that appear in Student's tasks and in the quizzes. In this way, mini-exams on one hand and quiz-supported open Student's sessions on the other, alternate during the course. Students are activated by mini exams, and provided extra opportunities to learn before them, partly via feedback to the teacher during the non-assessment phases and from quiz feedback. This alternation of examination and mathematical openness is a main underlying design feature.

After the course, in the written exam, the same knowledge is tested, but then requiring not only familiarity but also more complex calculations and combinations of several ideas in the same task.

The diagnostic quizzes are beneficial for a few different reasons. First, students refresh what they need to know to understand the lectures of the coming week. Typically, quiz-items will reappear during lectures since they are fundamental. Second, the teacher can, before the lecture, look at the number of correct answers on different questions and prepare to be more specific during the lecture on questions with many incorrect answers. The teacher thus obtains feedback from the student group with very little extra work.

Another reason not to assign points to Concept quizzes according to the number of correct answers is that they represent new facts that cannot be expected to be understood and examined yet. However, the students are informed that some quiz questions will appear at the final examination.

4. Comments about the realisation

Crucial for the result of this particular course design is the quality of teacher-student dialogue and interaction, since it is designed to inspire such interaction. From a course result point of view, most important is that feedback provides relevant and correct mathematics, but also that the feedback increases the student's interest and willingness to work, i.e. increase efficient student work time during the course weeks. This latter goal involves in turn many different aspects. I argue that fundamental here is the teacher's ability to listen: to find the real issue for the student. This issue may be trivial from the teacher's perspective, and if so impossible to spot for a teacher who does not try to take the student's perspective.

It is important for such feedback to communicate general properties of problem solving in mathematics. One such property is that mathematical problem solving typically is tentative: we try to use rules to change our expressions, but during calculation we cannot be sure if the calculation will actually lead to the result, nor what that result will be. This is normal, and part of the creative aspect of mathematical problem solving. This and other general observations about mathematical problem solving are rarely found in text books, and this is why it is even more important that it is communicated by the teacher. Classroom dialogue provides an excellent opportunity. Students may find a more realistic and practical view towards their own mathematical work.

Another general comment is that some errors during Student's sessions are probably similar to errors seen during grading of exams, but which a teacher perhaps never mentions during a lecture. Teacher attention and explanation to this is certainly helpful for some students.

A central experience during the realisation was that feedback to students in the students' session almost always can be done both mathematically accurately, relevantly, and in a way that is constructive and encouraging for students. This helps students' participation in the course. Below are some examples of such feedback:

- Student: Makes a common student error.

  Teacher: "*You are really not alone in this error. And it is very good that it appeared before the exam!*" (Followed by a full explanation.)

  Implicit conclusion: The student has indirectly helped other students in class.

- Student: Misses important argument in the calculation.

  Teacher: "*You may think this, but you need to write it down also, otherwise you lose points unnecessarily. When grading I can only consider what you have actually written down*".

  Implicit conclusion: It becomes clearer for a student how an exam is graded, and what needs to appear in a complete solution.

- Student: Hesitates significantly halfway in to an attempted solution.

  Teacher: "*You have made a good start, if you want I can finish?*"

  Implicit conclusion: A course is process towards knowledge, and it is only natural that the knowledge is not immediately complete.

For the sake of the listening students during a Student's session, it is important that the teacher also provides relevant feedback for them. The typical question: "*Are there other ways to solve this problem?*" after a student's solution may reveal the degree of problem solving activity that has been going on.

Access to quizzes can also be released in the end of the course after all quiz deadlines have passed, for repetition and practice. This was asked for by students.

Is there a large extra teacher workload for this design? No, not really. Once produced, the quizzes can be used for many years, with successive minor improvements. The main extra work is grading the mini-exams. Since mini-exams do not require long calculations, this grading is a matter of a few minutes per paper. Similarly, time required for the teacher to study the results on the quiz questions is not large. However, to make the first design of the course is time consuming, particularly to find good quiz statements that cover the essentials of the course, and explanations to them.

5. Measurable course results

In the written exam, 98% of the registered students participated, and 52% passed. The typical percentage pass rate for the Analys 2 course is around 40%. Colleagues regard the written exam as slightly more difficult than usual (certainly not easier). While at the student's course evaluation, the course was rated good or very good in all respects.

The university teacher, Johan Silvander, was present during one Friday lecture as a part of a university pedagogical project, and he evaluated the activity as follows:

"*A student from each group showed the solution of their specific exercise on the whiteboard. The teacher encourages students who have not been in front of the whiteboard before to show the solutions. The teacher is helping the students by putting leading questions, if needed, and explains to the audience what is taking place on the whiteboard. If some steps are not included by the student, these steps are filled in by the teacher at an appropriate time. The students in the audience were active and asked questions or proposed solutions. Questions and proposals that could have been considered as 'stupid' were handled in a way that enlightened the student and saved the student's*

face. Since the same basic methodology was used to solve all of the exercises the teacher pointed that out during all the exercises. The teacher took care of informing the students about its relevance by always showing all the steps of the methodology even if more advanced students have omitted a step due to 'This can easily be understood' ".

The take-aways from this session were:

- Letting the students show and explain their knowledge;
- Guiding the students to the answer and handling the students with respect;
- Using different abstraction levels when introducing the students to a new concept/methodology.

6. Conclusions and discussion

Mathematics is not an empirical area of research – it relies only on proof. However, a mathematics course is a highly empirical enterprise. Students' learning is the overall goal, and the quality of learning depends crucially on how well the design of lectures and activities match the prerequisite knowledge of the students that are registered to the course. This prerequisite student knowledge is the crucial empirical knowledge.

In order for a course to be scientific, a basic requirement is to collect relevant empirical knowledge, and use it to improve the result. I thus argue that for a course to be scientific it is not enough that the theory described is correct and relevant.

The course design presented in this paper aims at systematically collecting and using knowledge about students' prerequisite knowledge as well as challenging the student's thinking in mathematics, in order to reach further in mathematics learning.

Furthermore, it attempts to highlight types of mathematical knowledge that is important but rarely present in a traditional mathematical accounts. One important such type is how to find a workable idea to a mathematical solution of a problem, and to construct a solution from that idea. This is obviously fundamental for students during an examination and when working as an engineer, but not present in a calculus textbook that presents theory and correct solutions only. A correct solution usually does not contain a discussion of different possible solution methods, despite this being very important for mathematics problem solving.

The intention of the feedback is not only to learn about students' mathematical knowledge. It is also to reflect upon mathematical solutions, to communicate aspects of mathematical problem solving that is not easy to define in a clear way. This raises questions about mathematics teacher competencies and properties of mathematics that is outside mainstream mathematics teaching. Which kinds of creativity and mathematics knowledge enable students to formulate relevant basic ideas in order to construct mathematical solutions? How do mathematics teachers create a positive atmosphere so an exploratory discussion about mathematical topics takes place that problem solving students find relevant and engage in? How are such competencies educated?

An experience from this feedback course design is that it favours students that are engaged in the course, which is not always the most successful students. The feedback also tends to make the course more inspiring and rewarding to the teacher, which indirectly favours students' studies and results.

For a teacher, there is an easy test concerning empirical course knowledge. Before grading a written exam, it is possible and fruitful for a teacher to try to predict the mean number of points that the students receive for the different tasks in the exam. Slightly more advanced than this is to predict

which kinds of errors are most common and which will invoke the largest loss of points before grading, and subsequently check this during the grading. This is a scientific teacher attitude – taking advantage of existing empirical knowledge.

7. References

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