CASE STUDY

Maths lecturers in denial about their own maths practice? A case of teaching matrix operations to undergraduate students

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Abstract

This case study provides evidence of an apparent disparity in the way that certain mathematics topics are taught compared to the way that they are used in professional practice. In particular, we focus on the topic of matrices by comparing sources from published research articles against typical undergraduate textbooks and lecture notes. Our results show that the most important operation when using matrices in research is that of matrix multiplication, with 33 of the 40 publications which we surveyed utilising this as the most prominent operation and the remainder of the publications instead opting not to use matrix multiplication at all rather than offering weighting to alternative operations. This is in contrast to the way in which matrices are taught, with very few of these teaching sources highlighting that matrix multiplication is the most important operation for mathematicians. We discuss the implications of this discrepancy and offer an insight as to why it can be beneficial to consider the professional uses of such topics when teaching mathematics to undergraduate students.

Keywords: Matrices, higher education, research and teaching practice, educational material and media.

1. Introduction

We teach undergraduate mathematics and foundation-year mathematics, and for some time we felt vaguely uncomfortable about what seems to be an unnecessary gap between the maths used by academics and the maths taught by academics in some parts of undergraduate mathematics. Imagine a driving instructor in the UK who drives, obviously, on the left-hand side of the road. It would be preposterous to imagine that they will teach their pupils to drive on the right-hand side of the road, i.e., contrary to what they do themselves. However, our study presented below shows that this impossible example seems to illustrate what maths lecturers might sometimes do at universities; when they conduct research, they use mathematics efficiently and professionally, but simultaneously, they seem happy to teach mathematics in a way that instils a somewhat distorted view of professional mathematical practice in the students that they teach.

As we aimed to quantify our imprecise discomfort expressed in the previous paragraph, we were successful at locating one specific small area of mathematics on which we could zoom in and explore in detail, as described below.

2. Matrix operations

Matrix operations feature in a typical first-year university curriculum and in the Further Mathematics A/AS level within the UK. In a typical curriculum, matrix operations include addition and multiplication, and sometimes subtraction and ‘scalar multiplication’ (that is, multiplying a matrix by a number) also explicitly feature. For instance, Further Maths includes a section “*Add, subtract and multiply conformable matrices; multiply a matrix by a scalar*” (Cresswell, 2006). This is the area of the curriculum on which we concentrate. Of course, there are also other operations one can apply to matrices, most notably, inverting, but these operations are nearly always introduced in other sections, not together with the operations above, and so we do not consider them here. One half of our study consisted of inspecting 40 teaching publications (that is, 20 textbooks and 20 lecture notes) to see how matrix operations are introduced in them.

Our professional experience as mathematicians has led us to believe that in applications of matrices, multiplication of matrices is used much more widely than addition. Let us present two examples from different ends of the spectrum. Firstly, in the Further Maths textbook by Cresswell (2006), there are two applications of multiplication ("*successive transformations*" and "*solve three linear simultaneous equations in three variables by use of the inverse matrix*") but no applications of addition. Secondly, in the book on deep learning by Chollet (2017), the author states that "*deep neural networks [consist] mostly of many small matrix multiplications*". We felt that a similar picture can be observed in mathematicians' research outputs. As such, the second half of our study consisted of inspecting 40 recent research publications to see what place matrix multiplication occupies in them.

Thus, the questions we were asking were approximately as follows:

1. Is it true that in teaching, defining addition of matrices and multiplication of matrices are treated as topics of an equal importance?

2. Is it true that in applications of matrices (as demonstrated in research publications) multiplication of matrices is by far the most important matrix operation?

3. If the discrepancy described in the previous two questions exists, is it justified? And if it is not justified, what can be done?

3. Research outputs

Here is how we produced our data. We selected two medium-sized university mathematics departments, University of Essex and University of East Anglia, that publish research papers in a wide range of areas of mathematics. We then used staff web pages of each department to select researchers who, according to their online biography, were likely to have completed research using matrices in some way. Then we performed a search in Google Scholar using the keywords: '<First name> <last name> matrix' and chose results where 'matrix' was highlighted in the description of the search result. We scanned the paper by eye and recorded whether we agree with the statement "*In this publication, among other matrix constructions, matrix multiplication plays the most prominent role*". We also made a note of the total number of pages in the paper (excluding bibliography) and the approximate number of pages where matrix multiplication is used. Admittedly, due to a very wide range of mathematical research which we scanned, with various generalisations and applications, we had to treat matrix multiplication somewhat broadly, as 'an operation that resembled matrix multiplication appeared to be used'.

We saw that in 33 publications out of 40, matrix multiplication was most prominent, and in only 7 it was not. In the latter, 'multiplication-poor' publications, multiplication was shunned not in favour of other matrix operations, but because no matrix operations were used.

As to the number of pages on which matrix multiplication features in a research paper, the ratio is shown in Figure 1. The horizontal axis shows the proportion of the pages in the publication which uses matrix multiplication. The vertical axis shows the number of publications. As you can see, the histogram is heavily skewed, with most of the data towards the right-hand side of the distribution, and half of research publications using matrix multiplication on at least 80% of their pages.

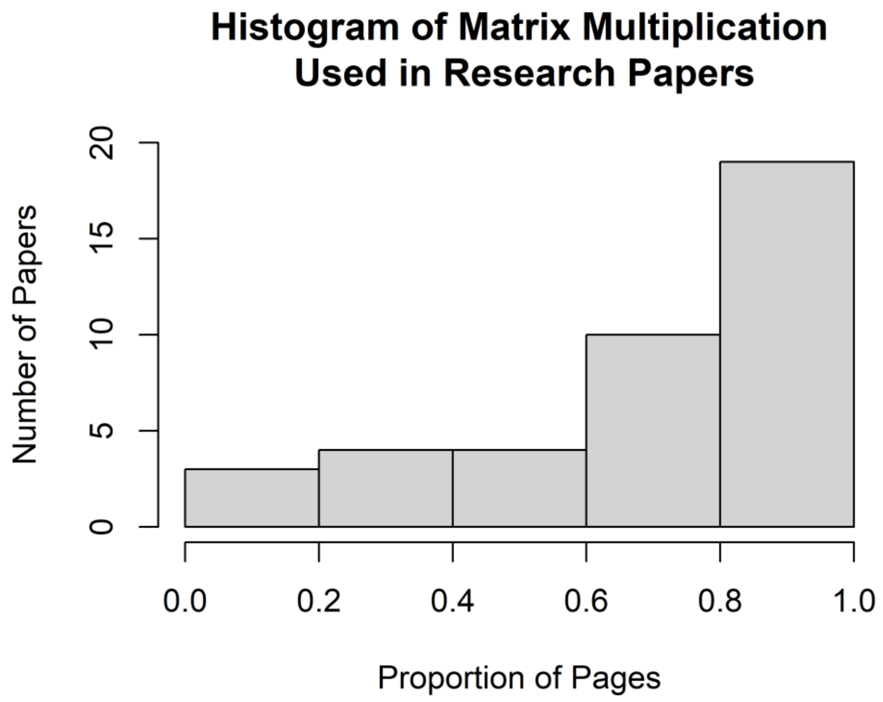

Figure 1. Proportion of pages in research papers that uses matrix multiplication as a prominent operation.

A preliminary conclusion from these observations is that among matrix constructions, matrix multiplication is of paramount importance in mathematical practice. In the next section we explore whether this fact is reflected in the way matrix operations are taught.

(Out of interest, we reflected whether not using matrix operations in some publications is a feature of these publications in particular, or of the research areas which they explore. In the publications we considered, the topics of 'multiplication-poor' publications are category theory, complex analysis, Markov chains, molecule imaging, and social interaction of animals. We can easily imagine that some other publications in these research areas could usefully employ matrix multiplication. Thus, not using matrix operations is a feature of specific publications.)

4. Textbooks and lecture notes

Here is how we produced our data. We selected textbooks and lecture notes from a range of years that were available as PDF documents online. We then used the contents page to locate the section where matrix operations were introduced. We scanned the section by eye and recorded whether we agree with the statement "*In the section on operations on matrices, most attention is paid to multiplication*". We made a note of the number of pages dedicated to matrix operations in total and to matrix multiplication in particular. In most cases, the textbook or lecture notes went on to explain applications of matrix operations, however we opted to remove these pages from our count and only include pages where matrix operations were first introduced to the reader.

We saw that in 24 textbooks and lecture notes out of 40, matrix multiplication was most prominent, and in 16 it was not.

As to the number of pages on which matrix multiplication features in a section on matrix operations, the ratio is shown in Figure 2, using the same bins as in the histogram in Figure 1. The horizontal axis shows the proportion of the pages in the publication which uses matrix multiplication. The vertical axis shows the number of publications. The histogram in Figure 2 is symmetrical, with both the mean and the median just under 70%.

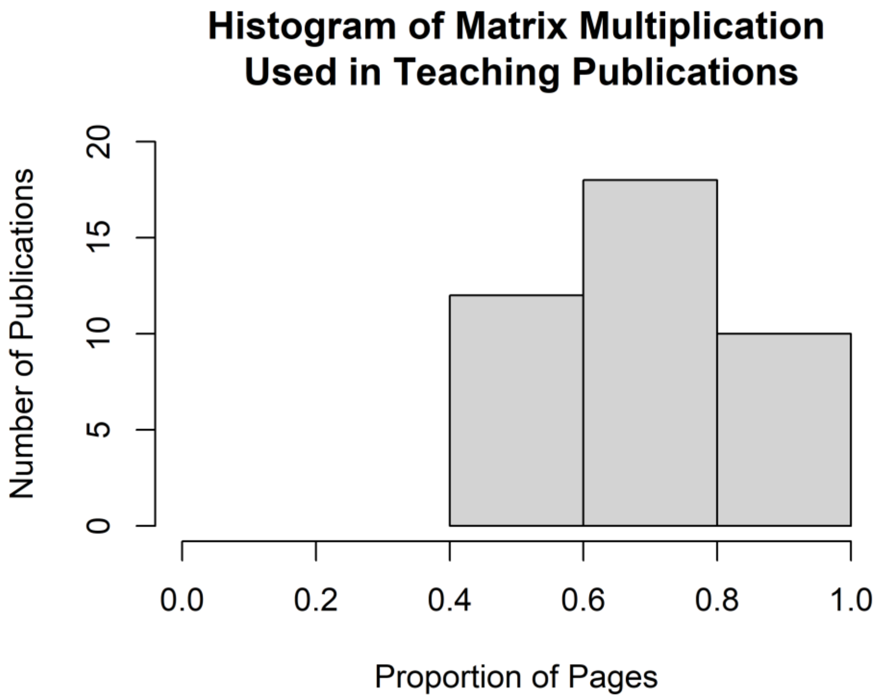

Figure 2. Proportion of pages in teaching publications that are used when introducing matrix operations and which are devoted to matrix multiplication.

In all sections on matrix operations matrix multiplication occupies more pages than other operations. If we pore over the text of these sections in more detail, the conclusions are mixed.

On the one hand, in some of the ‘multiplication-heavy’ sections, multiplication was given more attention not because it is presented as more important, but because its definition is perceived as being more complicated than those of other matrix operations. Some books explicitly suggest that out of the two operations, addition and multiplication, multiplication is the ‘uglier’ one. For example, in the textbook by Olver and Shakiban (2006), the definition of multiplication of matrices is immediately followed by saying “*Now, the bad news. Matrix multiplication is not commutative*”. In the textbook by Lang (2012), addition and scalar multiplication are parts of the definition of matrices, whereas multiplication is less so; indeed, the author defines not matrices, but “*the space of matrices*”. Similarly, for Bourbaki (1958), addition is more natural because addition of matrices can be defined for matrices over any additive group, whereas multiplication of matrices can be usefully defined only for matrices over an associative ring.

On the other hand, some textbooks and lecture notes skew the section on matrix operations towards multiplication in what seems a clear recognition of the more important role of multiplication. Out of the 40 textbooks and lecture notes, only 7 have 90% or more of the pages in the section on matrix operations dedicated to multiplication.

Hardly any textbooks or lecture notes explicitly state that matrix multiplication is more important. However, one example of a balanced solution is found in Birkhoff and MacLane (2017); first addition and scalar multiplication are introduced as "*vector operations on matrices*", and then matrix multiplication is introduced as "*the most important combination*" of matrices.

5. Discussion and conclusions

Does the comparison presented in the previous two sections matter? Let us explain why we undertook this study. We agree with Harari (2018) saying,

> the last thing a teacher needs to give her pupils is more information. They already have far too much of it. Instead, people need the ability to make sense of information, to tell the difference between what is important and what is unimportant, and above all to combine many bits of information into a broad picture of the world.

Not all undergraduate students will proceed to reading research papers and seeing which matrix operations are used or not used there. Millions of people might scan the lists of topics in A level subjects but never attempt these A levels. If we do not immediately present mathematical definitions and facts in the way in which they really are used in the practice of professional mathematicians, there might be no other opportunity. For many people, after they have read 'add, subtract and multiply matrices' in a mathematical curriculum, it will stay with them for life and slightly distort their mental image of mathematics. Somewhat exaggerating, to read 'add, subtract and multiply matrices' in a mathematical curriculum is like to read 'Chertsey, Upminster and London' in a geography curriculum; Chertsey and Upminster might be fine places, but they are less important than London and are not likely to feature in the same list with London and precede it.

When we teach matrices, we the authors grasp an opportunity to immediately show our own students that a clever definition of matrix multiplication makes this operation versatile and usable in many applications, and that this definition alone makes matrices usable in many applications. Reflecting on our observations presented in the previous sections, we eventually migrated towards a practice when we introduce matrix multiplication as an important construction, and in the meantime define matrices as notation which is convenient to use when one performs matrix multiplication. An example of one of activities that we use is given below:

A chelsea bun contains 45 grams of flour, 5 grams of sugar, 15 grams of milk and 1/10 of an egg. A brioche bun contains 45 grams of flour, 2 grams of sugar, 2 grams of milk and 2/10 of an egg. In 100 grams of flour there are 16 grams of protein, 86 grams of carbohydrate and 3 grams of fat. In 100 grams of sugar there are 100 grams of carbohydrate. In 100 grams of milk there are 3 grams of protein, 5 grams of carbohydrate and 1 gram of fat. In one egg there are 7 grams of protein and 5 grams of fat. I ate one brioche bun and two chelsea buns.

Express all the data from the previous paragraph as matrices. Multiply these matrices to calculate how much protein, carbohydrate and fat I consumed.

In addition to the example given above, you can see the first author, Alex Partner, introducing another activity with a toy food-based example in his video lecture (Partner, 2020). Our approach is similar to that of Dunn and Parberry (2002), where multiplication is described as the only interesting operation to be performed, particularly from the perspective of linear transformations. Eddie Woo uses a similar approach in one of his videos, also employing a toy food-based example (Woo, 2014).

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