MSOR connections

Articles, case studies and opinion pieces relating to innovative learning, teaching, assessment and support in Mathematics, Statistics and Operational Research in HE.

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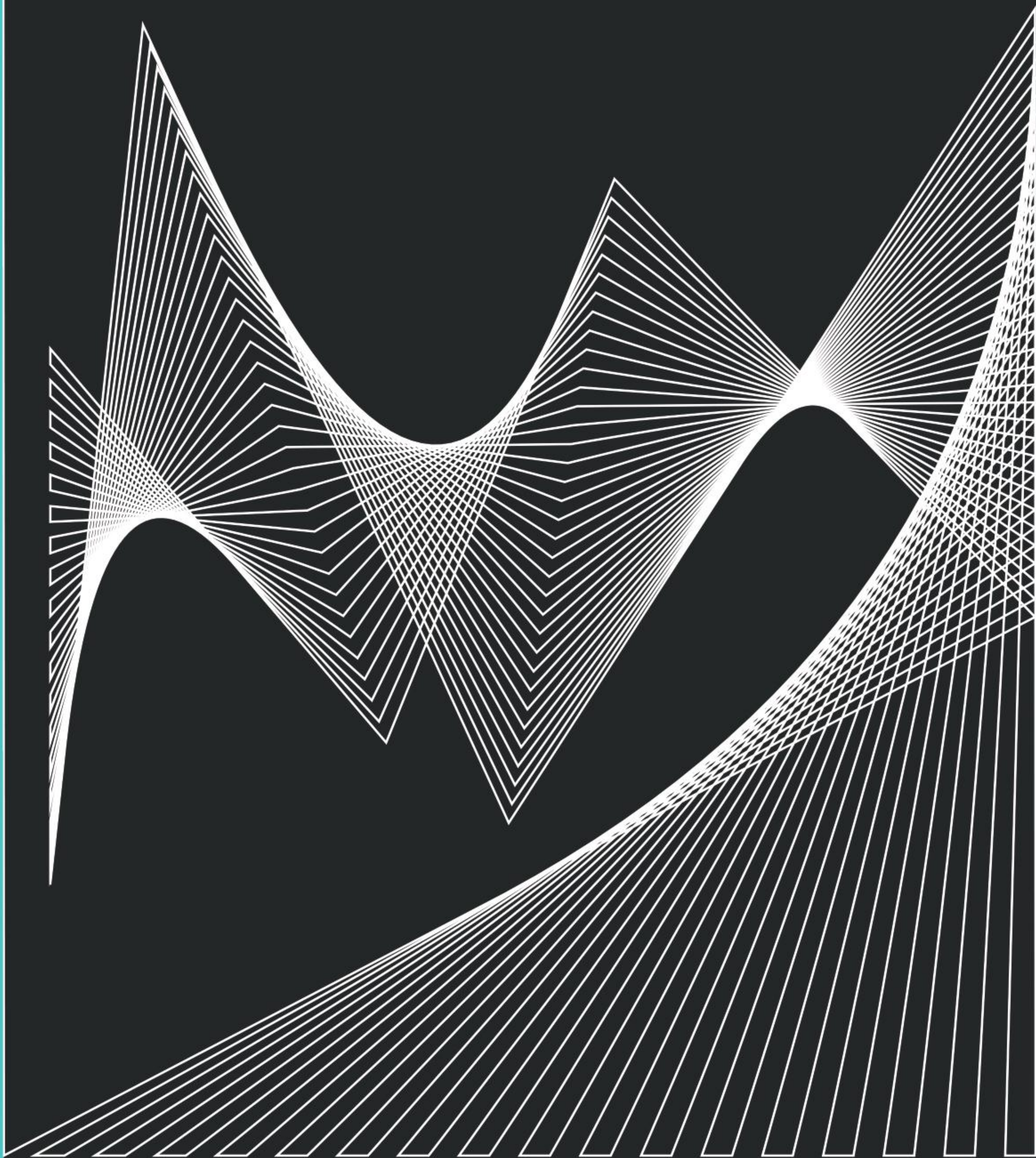

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Editorial

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Welcome to this third issue of Volume 15 of *MSOR Connections*. As usual, this contains a varied mix of research articles, case studies, opinion pieces and resource reviews, all with very relevant and immediate applications to teaching in the MSOR subject area. These cover topics related to the mathematics A/AS level reforms, mathematics outreach, maths anxiety, motivating mature students to sudy mathematics and international mathematics competions, but we also have some ‘doodling’, some ‘knitting’ and even some football!

We ‘kick off’ with two articles that focus on students’ experience in pre-university mathematics. The first by Cronin, Shuilleabhain, Lewanowski-Breen and Kennedy, examines the impact of participating in a series of mathematics workshops called ‘Maths Sparks’ on secondary-school pupils’ attitudes towards mathematics. This is then followed by a piece by Paul Glaister, which provides a detailed update of the current reforms to AS and A levels in Mathematics and Further Mathematics, with links to relevant sources of information and resources and asks “are you ready?”.

Ending the ‘first-half’ of this issue is another pair of excellent articles that examine issues related to anxiety and motivation in the study of mathematics. First up are Marshall, Staddon, Wilson and Mann who discuss some of the strategies, implemented at the University of Sheffield, to reduce anxiety and engage students in the learning of university mathematics. The second, by Mulligan and Mac an Bhaird, focuses on challenges that mature students face in their mathematics education when studying for a pre-degree Certificate in Science at a university in Ireland.

Beginning the ‘second half’ of this issue is a pair of articles, which focus on classroom practice. The first of these, by McLoone, Kelly and NiShe, presents a novel multi-platform smart device-based student response system, called UniDoodle (a development of the electronic voting systems or ‘clickers’). Nicola Reeve then follows this with a resource review of the ‘knitr’ package, for use with the R statistical programming environment. This illustrates how to ‘knit’ together R and LaTeX code, so that the R code and output and the narrative are all included in one source document to make the process of producing course materials for teaching statistics with R very efficient indeed.

Bringing us towards the ‘final whistle’ is an an article by Phil Scarf who shares with us 30 years of final year projects in Mathematics, Statistics and Operational Research, with numerous ideas for using sport as the basis for such projects. The ‘close of play’ and ‘post-match interview’ is then provided by the final article of this issue where we find out that Wang and Xu and a group of students at Coventry Univeristy are the eventual winners, as they share their experience with us of taking part in a multi-day international mathematics competition held annually in the USA.

Please do continue though to write up your own work within the teaching and learning of the MSOR subjects so that we can continue to share your good practice with others. As always, I would like to thank my fellow editors, the editorial board and all reviewers for their support in preparing this issue. To register for submissions/notifications, and for further information relating to *MSOR Connections* please visit https://journals.gre.ac.uk/index.php/msor.

RESEARCH ARTICLE

Maths Sparks: Investigating the impact of outreach on pupil's attitudes towards mathematics

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Abstract

In this article, we examine the impact of participating in a series of mathematics workshops on secondary-school pupils' attitudes towards mathematics. A six-week program, entitled 'Maths Sparks', was run by a team of lecturers and students at a research-intensive university in the Republic of Ireland. The outreach series aimed to promote mathematics to pupils from schools designated as socio-economically disadvantaged (DEIS - Delivering Equality of Opportunity in Schools), who are less likely to study mathematics at higher level than their non-DEIS counterparts (Smyth et al. 2015). Sixty-two pupils participated in the research and data was generated through pre-post questionnaires based on the Fennema-Sherman (1976) framework of Attitudes to Mathematics. Findings suggest that while male students initially had more positive attitudes towards mathematics, there was a narrowing in this gender gap across several factors on the Fennema-Sherman scale as a result of participation in the programme. The most prominent of these features were: 'Attitudes towards success in mathematics' and 'Motivation towards mathematics'. Findings suggest that the construct and delivery of this Mathematics outreach programme, involving undergraduate students and academic staff, may provide a useful structure in benefitting pupils' attitudes towards mathematics and encouraging their study of the subject.

Keywords: mathematics outreach, widening participation, student-staff collaboration.

1. Introduction

Students from lower socio-economic areas are often disadvantaged in terms of their mathematics education, when compared with their counterparts in higher socio-economic areas (Cox & Bidgood, 2002; Schoenfeld, 2002). In the Republic of Ireland (ROI), pupils in schools designated as disadvantaged (Delivering Equality of Opportunity in Schools – DEIS) have been found to have lower average scores in overall mathematical skills (Shiel, Kelleher, McKeown, & Denner, 2015). In addition, a recent report has found that secondary pupils in DEIS schools are less likely to participate in mathematics at a higher level than their non-DEIS counterparts (Smyth, McCoy, & Kingston, 2015). It has been suggested that this lower participation in higher mathematics is due to pupils' attitudes towards studying mathematics and their perceived usefulness of studying mathematics at higher level for the terminal state secondary examination (the Leaving Certificate).

In this paper, we investigate the impact on pupils' attitudes towards mathematics via participation in a series of mathematics outreach workshops, known as 'Maths Sparks'. Building on results from a previous pilot programme (Ni Shuilleabhain & Cronin, 2015), this research utilizes the Fennema-Sherman framework of attitudes towards mathematics (1976) to analyse the impact of participation in this series of workshops on pupils' attitudes towards mathematics. In addition, we analyse the impact on pupils' aspirations to pursue higher level mathematics as a result of taking part in Maths Sparks.

2. Maths Sparks

'Maths Sparks' is a series of mathematical problem solving workshops, based on content external to the senior-cycle mathematics curriculum. Each workshop is designed to incorporate contextualized and meaningful mathematical activities which encourage students' sense-making and include a range of classroom organizational forms (Schoenfeld, 1992; Verschaffel et al., 1999). A key feature of the workshops is that they are designed by teams of undergraduate students, who collaborate with academics to develop relevant mathematical content for participating pupils. Each workshop is presented by the designing undergraduate team, with pupil learning facilitated by other participating students and lecturers. This construct encourages collaboration between secondary pupils and undergraduate mathematics students in exploring mathematical ideas and developing skills in mathematical thinking.

The workshops are designed with four aims:

1. To encourage pupils to communicate, reflect on, and build confidence in their mathematical thinking.
2. To impact pupils' attitudes towards mathematics to see it as a viable, interesting, and important subject.
3. To motivate pupils to continue studying mathematics at higher level for Leaving Certificate.
4. To encourage pupils to study courses related to science, technology, engineering or mathematics (STEM) at third level through engaging with undergraduates and graduates of STEM.

The series is aimed towards senior cycle secondary pupils in Transition Year or fifth year (15-17 year olds) and is offered free of charge to participants.

Information on the Maths Sparks series was shared with DEIS schools through the university Access & Lifelong Learning Centre, who have responsibility for providing support to school leavers from socio-economically disadvantaged backgrounds. Seventy-two pupils (forty-one females and thirty-one males) from twelve schools chose to take part in the programme.

The series of workshops took place from March to April in 2016 and introduced 10 topics (from Game Theory to Cryptography) over the course of the six weeks. Workshops were held in the university's Active Learning Rooms (ALE), where the learning environment supported collaborative activities for pupils. On each of the six evenings, a short concluding presentation was given by a mathematics lecturer on a topic related to the workshops.

3. Methodology

Prior to the commencement of the series, pupils were invited to be involved in research on the impact of taking part in Maths Sparks. Permission for pupils' participation was requested from parents/guardians and sixty-two pupils (thirty-seven females and twenty-five males) agreed to take part. Data was generated through pre-post questionnaires, which contained a mixture of open and Likert-scale questions. Open questions were designed to investigate pupils' opinions on studying mathematics and their intentions to study mathematics at higher level for the Leaving Certificate. Likert scale questions on a 5-point scale ranging from "strongly disagree" to "strongly agree" were asked on 12 questions over six of the nine factors of the Fennema-Sherman scale (1976):

1. Mathematics Anxiety (MA),
2. Confidence in Learning Mathematics (CLM),
3. Attitudes Towards Success in Mathematics (ATS),
4. Teacher Scale (TS),
5. Usefulness of Mathematics (UM),
6. Effectance Motivation in Mathematics factor (EMM).

In total, sixty-two pupils (37 females and 25 males) completed the pre-series questionnaire and fifty pupils (23 females and 27 males) completed the post-series questionnaire. Statistical analysis incorporated only matching pre-post pupil responses (forty-four pupils) and qualitative analysis was conducted through a thematic analysis (Braun & Clarke, 2006) of pupils' responses based on the framework of the six Fennema-Sherman factors (1976).

4. Findings

As might be expected from pupils who had opted to participate in a mathematics outreach programme, in the pre-series questionnaires the majority of pupils were positive in their attitudes towards mathematics (all names are pseudonyms):

> *"I like the satisfaction that comes with it as soon as you solve the puzzle it feels very rewarding to me and it makes me feel that I have achieved my objective."* – Cora

> *"I like the challenge associated with it and the feeling of satisfaction when I solve parts of it."* – Sean

Participants' negative opinions of mathematics were, however, generally related to their experiences of learning mathematics at school. Aligning with research from (Lyons, Lynch, Close, Sheerin, & Boland, 2003) on classroom mathematics practices in the ROI, pupils reported on learning rules and formulae, with memorization viewed as an important mathematical skill. In addition, mathematics was often viewed as a topic solely relevant to school.

> *"I dislike all the theorems and things that need to be memorized for the Leaving Cert."* – Lucy

> *"Some chapters can be tedious. Just putting different numbers into the same formula repeatedly."* – Michael

When asked if they intended to study mathematics at higher level for their Leaving Certificate six pupils (all female) reported in the negative and cited a lack of confidence in their own ability as a reason for this decision.

> "*Because I am not good enough."* – Rachel

Following their participation in the Maths Sparks series of workshops, pupils' were asked if they felt more confident in their mathematical ability, with the vast majority of pupils responding positively.

> "*Yes, because I'm no longer afraid answering questions*" – Paul

> "Y*es, I give up less easily when tackling Maths problems and I see the problems through - attempt them to the best of my ability*" - Nicholas

All pupils in the post-series questionnaires intended to pursue mathematics at higher level in their Leaving Certificate and, while thirteen pupils (12 females and 1 male) in the pre-series questionnaire noted they were not considering a STEM related course or career after secondary school, in the post-series questionnaire this was reduced to five pupils (3 of whom were female). Pupils also had a broader perspective on the usefulness of mathematics across a variety of applications and careers:

> "*I've discovered several different things the students studied in relation to maths that interested me that I didn't even know I could study.*" – Ciara

> "*My opinion of maths has greatly changed as I thought it was only used in school and business but I later found out it could be used in game mechanics and measuring waves.*" - John

The quantitative analysis of the data further explores the impact of the Maths Sparks series on pupils' attitudes towards mathematics. Following pupils' participation in Maths Sparks, there were statistically significant results over three features of the Fennema-Sherman scale (1976): 'attitudes towards success in mathematics', 'usefulness of mathematics' and 'effectance motivation in mathematics' scales (results are included in the Appendix). Differences in pupils' responses across the two genders were evident, with male pupils' responses differing to those of female pupils in both pre- and post-series questionnaires and male pupils demonstrating a statistically significant change in their 'confidence in learning mathematics' post series. Differences in gendered responses were, however, reduced across 'attitudes towards success' and 'effectance motivation': male responses were significantly higher at the 5% and 10% level in the pre-survey (p-values for the Mann-Whitney U test 0.0151 and 0.06832 respectively) and these differences were no longer significant in the post survey.

Taking one of the features demonstrating a statistically significant change, we consider one of the twelve questions related towards 'attitudes towards success in mathematics'. Pupils were asked to rate their agreement with the statement "I don't like people to think I'm smart at maths". While in the pupils' pre-series responses thirty pupils disagreed or strongly disagreed with this statement, this increased to thirty-six in the post-series response. This change was related to the additional female pupils who strongly disagreed with this statement in the post-series questionnaire.

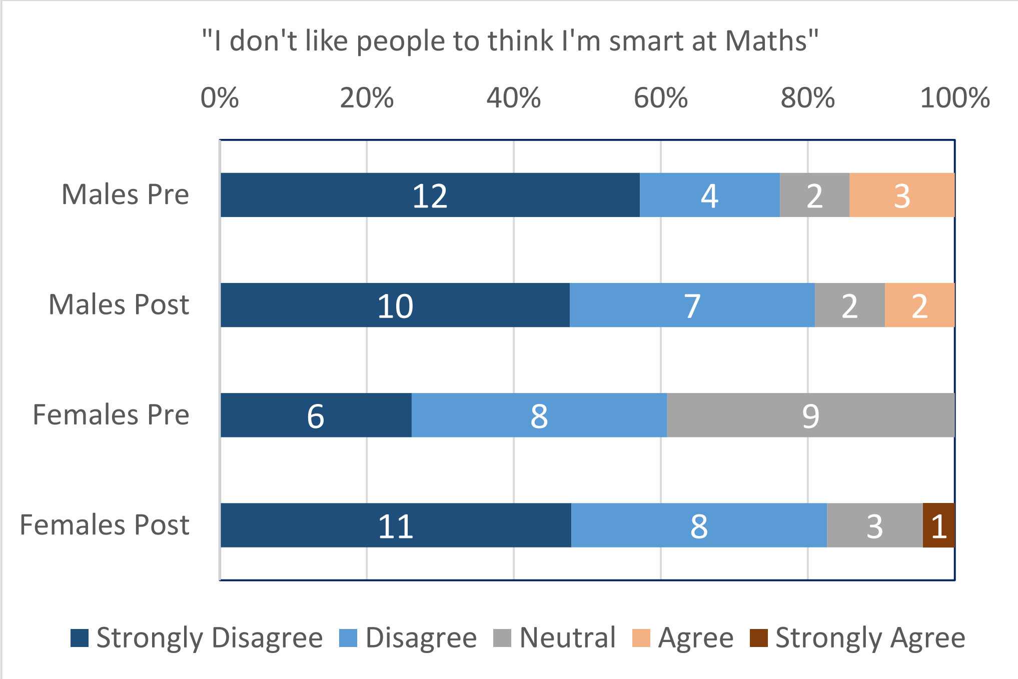

Figure 1. Responses to the question “I don’t like people to think I’m smart at maths.”

While only male pupils’ responses demonstrated a significant change in their ‘confidence in learning mathematics’, there was an increase in the numbers of pupils who strongly disagreed with the statement “For some reason even though I study, maths seems unusually hard for me”.

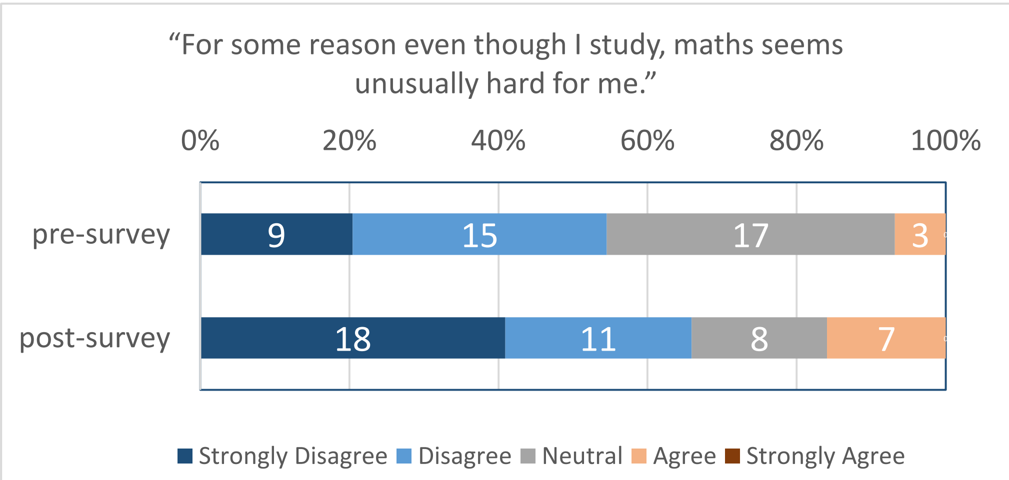

Figure 2. Negatively phrased question from CLM factor showing increase in positivity for the whole cohort

There were also increased numbers of pupils who agreed with the statement “I think I could handle more difficult maths” (further details included in the Appendix).

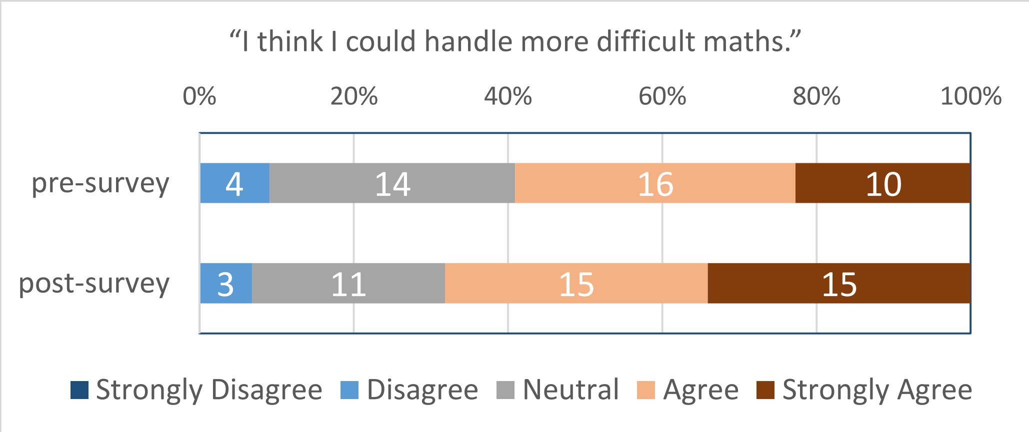

Figure 3: CLM questions showing increased positivity for the whole cohort in the post survey

Returning to pupils' open responses, participants were asked if their opinion of mathematics had changed due to their participation in Maths Sparks. Their responses were very positive and, in the majority, were related to their perceived relevance of the subject:

> *"It made me admire mathematicians and maths because of how much importance it has in our world."* - David

In addition, many pupils commented on the contrasting way mathematics was introduced in the Maths Sparks workshops when compared with their classroom experiences. Pupils also enjoyed learning new topics outside of the mathematics curriculum.

> *"Learning the different things to maths. In school, it's just "Find XYZ", here there are millions of different ways to show maths."* - Janette

> *"Yes. I learned that there are different options in maths. It's not like the maths we do in school. You can take the part in maths that you enjoy and find interesting to study."* - Marion

Many pupils also commented on being more aware of their capacity to work through problems with different strategies, rather than only having 'one way' to do a question:

> *"I know that I have to look at every question in other ways to get an answer*" – Cora

When asked to describe what they enjoyed about Maths Sparks, pupils commented on the social element of the workshops, where pupils engaged with undergraduate mathematics students and met and worked with pupils from other schools.

> *"Interacting with University students and getting to ask them questions."- Nicolas*

> *"Got to meet and befriend people I wouldn't normally have got the chance to meet."- Karen*

Pupils responded positively to both the content and construct of the Maths Sparks workshops:

> *"I really really enjoyed the maths we learned and now enjoy maths again. Thank you for doing this for us and offering it to us, it was very enjoyable"*- Michelle

> *"It was the best 8 weeks of my school life"*- Greg

5. Discussion and Conclusion

Participating in Maths Sparks workshops positively impacted on pupils' attitudes towards mathematics and on their intention to pursue mathematics at higher level in their secondary school studies. Analysis demonstrates that there were statistically significant changes to pupils' 'attitudes towards success in mathematics', 'usefulness of mathematics' and 'effectance motivation in mathematics' features of the Fennema-Sherman (1976) attitudes towards mathematics instrument. In addition, there were statistically significant differences in male pupils' responses to their 'confidence in learning mathematics'. Pupils' perceptions of mathematics were impacted and, rather than seeing mathematics as a subject within the school curriculum, pupils' responses included the references to applications of mathematics in the real world and across a number of industries.

Following their participation in the Maths Sparks series, more pupils intended to study mathematics at higher level in the Leaving Certificate and more pupils were considering pursuing a career in STEM.

Based on our findings, we consider that Marks Sparks offers the university an innovative way to attract students from a diverse socio-economic background to mathematics-based courses.

6. Acknowledgements

We like to thank the undergraduate volunteers and the secondary students who participated in Maths Sparks 2016 and would also like to acknowledge our undergraduate co-authors who supported the research as part of a summer internship programme. We would also like to acknowledge Áine Murphy and the UCD Access and Lifelong Learning Centre for their support in liaising with all schools participating in the programme.

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A booklet detailing each of the Maths Sparks workshops has been published and is free to download at https://www.ucd.ie/mathstat/mathsparks/ for any group wishing to commence a Maths Sparks programme at their own institution.

7. Appendix: Tables and Charts

Table 1. Summary of testing differences in the median results between and within genders

	Test	Result	p-value	U-Value	Z-value
MA	Males vs Females Pre Survey	Males had a higher median score on average	0.01169*	111	-2.5039
	Males vs Females Post Survey	Males had a higher median score on average	0.02184*	103	-2.0703
	Male Pre vs Male Post	No Significant Difference	0.625	2.5	-1
	Female Pre vs Female Post	No Significant Difference	0.6875	7	-0.8165
CLM	Males vs Females Pre Survey	Males had a higher median score on average	0.001851**	117	-3.0211
	Males vs Females Post Survey	Males had a higher median score on average	0.001111**	119	-3.1333
	Male Pre vs Male Post	Confidence was statistically significantly higher in the post survey	0.03125*	0	-2.2361
	Female Pre vs Female Post	No Significant Difference	0.625	2.5	-1
ATS	Males vs Females Pre Survey	Males had a higher median score on average	0.0151**	106	-2.1835
	Males vs Females Post Survey	No Significant Difference	1	73.5	-0.0982
	Male Pre vs Male Post	No Significant Difference	1	2	-0.5774
	Female Pre vs Female Post	No Significant Difference	0.125	0	-1.7321
TS	Males vs Females Pre Survey	Males had a higher median score on average	0.00062**	105	-2.2958
	Males vs Females Post Survey	Males had a higher median score on average	0.01015*	103.5	-2.171
	Male Pre vs Male Post	No Significant Difference	0.1875	12	1.3416
	Female Pre vs Female Post	No Significant Difference	0.25	3	1.4142
UM	Males vs Females Pre Survey	No Significant Difference	0.155	96	-1.8459
	Males vs Females Post Survey	Males had a higher median score on average	0.04535*	106	-2.1319
	Male Pre vs Male Post	No Significant Difference	0.5	0	-1.4142
	Female Pre vs Female Post	No Significant Difference	0.25	0	-1.7321
EMM	Males vs Females Pre Survey	Males had a higher median score on average	0.06832*	94	-1.9579
	Males vs Females Post Survey	No Significant Difference	[-]		
	Male Pre vs Male Post	No Significant Difference	0.125	6	1.7321
	Female Pre vs Female Post	No Significant Difference	0.5	0	-1.4142

* Significant at the 5% level
** Significant at the 1% level
[-] This test did not give a result as the median responses for males and females were *identical* leading to an NA value for the p-value, the result here is simply that there is zero difference between male and female responses for Effectance Motivation in the post survey

Table 2. Statistically significant results from testing differences between genders

Factor	Test	Result	p-value
ATS	Males Pre vs. Females Pre	Males had a significantly higher median response	0.0151**
	Males Post vs. Females Post	No Significant Difference between the genders	~ 1
EMM	Males Pre vs. Females Pre	Males had a significantly higher median response	0.06832*
	Males Post vs. Females Post	No Significant Difference between the genders	~ 1
UM	Males Pre vs. Females Pre	No Significant Difference between the genders	0.155
	Males Post vs. Females Post	Males had a significantly higher median response	0.04352**

** significant at the 5% level * significant at the 10% level

Table 3. Results showing males having much higher results in several factors, indicating a gender divide.

actor	Test	p-value	Result
MA	Males vs Females Pre Survey	0.01169	Males had a higher median score on average
	Males vs Females Post Survey	0.02184	Males had a higher median score on average
CLM	Males vs Females Pre Survey	0.001851	Males had a higher median score on average
	Males vs Females Post Survey	0.001111	Males had a higher median score on average
TS	Males vs Females Pre Survey	0.00062	Males had a higher median score on average
	Males vs Females Post Survey	0.01015	Males had a higher median score on average

8. References

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology, 3*(2), 77-101. doi:10.1191/1478088706qp063oa

Cox, B., & Bidgood, P. (2002). Widening Participation in MSOR. *MSOR Connections, 2*(1), pp.15-19.

Fennema, E., & Sherman, J. A. (1976). Fennema-Sherman Mathematics Attitudes Scales: Instruments Designed to Measure Attitudes toward the Learning of Mathematics by Females and Males. *Journal for Research in Mathematics Education, 7*(5), pp. 324-326. doi:10.2307/748467

Looney, A. (2006). Assessment in the Republic of Ireland. *Assessment in Education: Principles, Policy & Practice, 13*(3), pp. 345-353. doi:10.1080/09695940601035544

Lyons, M., Lynch, K., Close, S., Sheerin, E., & Boland, P. (2003). *Inside Classrooms: The Teaching and Learning of Mathematics in Social Context*. Dublin: Insitute of Public Administration.

Ni Shuilleabhain, A., & Cronin, A. (2015). Maths Sparks: Developing Community and Widening Participation. *MSOR Connections, 14*(1), pp. 43-53.

Schoenfeld, A. H. (1992). Learning to think Mathematically: Problem Solving, Metacognition, And Sense-Making in Mathematics. In D. Grouws (Ed.), *Handbook for Research on Mathematics Teaching and Learning* (pp. 334-370). New York: MacMilan.

Schoenfeld, A. H. (2002). Making Mathematics Work for All Children: Issues of Standards, Testing, and Equity. *Educational Researcher, 31*(1), pp. 13-25.

Shiel, G., Kelleher, C., McKeown, C., & Denner, S. (2015). *Future Ready? The performance of 15-year-olds in Ireland on Science, Reading Literacy and Mathematics in PISA 2015*. Retrieved from Dublin.

Smyth, E., McCoy, S., & Kingston, G. (2015). *Learning from the Evaluation of DEIS*. Retrieved from Dublin.

Verschaffel, L., De Corte, E., Lasure, S., Van Vaerenbergh, G., Bogaerts, H., & Ratinckx, E. (1999). Learning to Solve Mathematical Application Problems: A Design Experiment with Fifth Graders. *Mathematical Thinking and Learning, 1*(3), pp. 195-229.

SHORT UPDATE

AS and A levels in Mathematics and Further Mathematics are changing - are you ready?

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Abstract

The purpose of this article is to give an overview of the reforms to AS and A levels in Mathematics and Further Mathematics, with links to relevant sources of information and resources, to assist colleagues in HEIs with their planning for curricula etc for new entrants in Autumn 2019. The article begins with a brief history of the reforms and their rationale, together with details of the final subject content and nature of the assessment regime in which these sit. The specific details of the eight major changes to Mathematics and Further Mathematics AS and A levels are included. The article concludes with some details on the related specifications and assessment materials provided by the Awarding Oganisations: AQA, OCR, and Pearson/Edexcel.

Keywords: mathematics education, AS/A level reforms, mathematics, further mathematics.

1. AS/A level reforms: history, timeline and rationale

The most recent reform of AS and A levels in Mathematics and Further Mathematics, with first teaching of the new qualifications about start in September 2017, began as far back at 2010 with the publication by the Department for Education (DfE) of the Schools White Paper 'The Importance of Teaching' (DfE, 2010), stating that:

> 'A levels are a crucial way that universities select candidates for their courses, so it is important that these qualifications meet the needs of higher education institutions. To ensure that they support progression to further education, higher education or employment, we are working with Ofqual, the awarding organisations and higher education institutions to ensure universities and learned bodies can be fully involved in their development.'

It wasn't until 2013 that any significant progress began, following the Secretary of State for Education's letter (DfE, 2013a) to the Chief Executive of The Office of Qualifications and Examinations Regulation (Ofqual) saying:

> 'I believe that the primary purpose of A levels is to prepare students for degree-level study. All students should have access to qualifications that are highly respected and valued by leading universities. Current A levels do not always provide the solid foundation that students need to prepare them for degree-level study and for vocational education. The modular nature of the qualification and repeated assessment windows have contributed to many students not developing deep understanding or the necessary skills to make connections between topics. Many leading universities are concerned about current A levels, and nearly three-quarters of lecturers say that they have had to adapt their teaching approaches for underprepared first year undergraduates. As you have found, there is support for much greater higher education involvement in A levels.
>
> There is clear dissatisfaction among leading university academics about the preparation of A level pupils for advanced studies. Mathematicians are concerned that current A level questions

> are overly structured and encourage a formulaic approach, instead of using more open-ended questions that require advanced problem-solving.
>
> It is of paramount importance that new A levels command the respect of leading universities. I am delighted that the Russell Group is planning to create an organisation to provide advice to Ofqual on the content of A levels. The advisory body will focus on those A levels which are most commonly required for entry to our leading universities and will seek the views of universities outside of the Russell Group, as well as engaging with relevant learned societies and others. The involvement of respected academics will help to ensure that the qualifications are designed to equip students for university. It will be critical that new qualifications are reviewed each year to ensure that they are delivering the rigorous and high quality education that is needed.'

As a consequence, all AS and A levels have been included as part of the reform process, with consequential changes in subject content, assessment, and particularly the assessment regime, including a change to 'decoupling' AS from A levels, and making A levels 'linear' (see Section 4 for further details). In three subject areas: mathematics, further mathematics; modern foreign and classical languages; and geography, these reforms have been more wide-ranging, and involved a much lengthier and detailed review, with an A Level Content Advisory Board (ALCAB) for each of these (ALCAB, 2013). (See also Smith, 2014, relating to recommendations for review of AS and A level subjects.) The A Level Content Advisory Board (ALCAB) for mathematics and further mathematics was then set up by the Russell Group in June 2013 (ALCAB, 2013) with support from the DfE and Ofqual (DfE, 2013b,c,d,e) to:

> '… advise on subject content … and play a lead role in an annual post-A level review'.

Following six months of intensive work during the first half of 2014, ALCAB's first report was published (ALCAB, 2014a), along with an executive summary (ALCAB, 2014b), comprising details of ALCAB's work and recommendations. Additional advice was published at the same time in a letter from Professor Richard Craster, Chair of the Mathematics and Further Mathematics ALCAB Panel (ALCAB, 2014c), along with the highlighting of some issues that needed further consideration on:

> '**Continuing Professional Development**. The suggested changes to content and style of A level mathematics may present challenges to existing teachers of mathematics and we strongly advise that continuing professional development courses in mathematics are adequately resourced so as to ensure that all teachers are equipped with the skills they need. The Further Mathematics Support Programme (FMSP) has done a magnificent job: the situation would have been far worse without its influence, and it is important that it is both supported and extended.
>
> **Monitoring and future development**. The panel views the continued scrutiny of A level Mathematics as essential in order to prevent a recurrence of the problems highlighted in our main report and to see through the implementation of these proposals. It is also important to allow examinations to develop in response to technological changes and also to developments in the subject itself. It is not desirable to have content fossilised at this point in time. There is therefore a need for continuing development to refine and improve the specifications and assessment. There is, for instance, value in having at least one developmental A level specification which has more innovative approaches to content and assessment and tests pedagogy that can later become mainstream (more embedded use of technology, discrete

> mathematics, etc). I have noted that the recently-published Royal Society "Vision for science and mathematics education" (RS, 2014) states that "new, independent, expert bodies that draw on the wider STEM professional community need to be created in England and Wales to determine curricula and assessment in STEM subjects". The ALCAB panel which I have chaired would like to see arrangements of that kind made for mathematics, as in our view it is essential for the matters raised in this letter to be kept under continuous review.'

The Royal Society's Advisory Committee on Mathematics Education (ACME) advises on '3-19 mathematics education policy in England', and with recent changes being made to ACME, (ACME, 2017a,b,c), the Committee will be well-placed to continue to provide this much-needed expert, independent advice to Government.

The final advice ahead of publication of the Subject Content for Mathematics and Further Mathematics can be found in (ALCAB, 2014d,e), with the response from the Secretary of State (ALCAB, 2014f) stating:

> 'I share your desire to see that ALCAB's intentions for these subjects are followed through into practice, generating significant rewards for students and others, not least universities.
>
> ALCAB's work has represented a major part of our commitment to working with universities on A level reform, and we will continue to pursue this commitment with some vigour.'

2. Reformed Subject Content and Assessment

ALCAB's recommendations for Subject Content were accepted in full by the DfE in December 2014, and published at that time (DfE, 2014a,b). It is only over the last few months, however, that teachers in schools and colleges have been finding out more about the implications for teaching and learning, and, of course, assessment. This might be of concern given that the direction of travel was made clear in the Subject Content documents (DfE, 2014a,b), but there is good reason for this. These implications would only become truly apparent once the associated Specifications and Sample Assessment Materials (SAMs) offered by the various Awarding Organisations (AOs, and also known as 'examination boards') had been approved by Ofqual as part of their accreditation process. For a variety of reasons it is only very recently that the AOs have had their Specifications and SAMs accredited, which now gives teachers a much fuller picture of what is expected, and for this reason it is only now that, for many teachers, the intentions and implications of the reforms are being fully understood.

As we are all too well aware in HE, whether we like it or not, assessment drives much of teaching and learning, and in schools and colleges the accountability measures are such that this is inevitable there too, not to mention the pressure on learners to perform well to be able to gain entry to their choice of university and programme. A significant amount of CPD is needed to realise these reforms, and much of this is being provided by the FMSP (FMSP, 2017a,b), often in partnership with the Maths Hubs (Maths Hubs, 2017).

3. Implications for HE

If one asks lecturers in HEIs the question in the title, however, there will be mixed responses, but the response will more than likely be something along the lines: "Changes? What changes?". Given that it will not be until Autumn 2019 when HEIs will admit the first group of students who have studied the reformed AS/A levels, there is ample time for teaching staff to find out more about the reforms, consider the implications for their programmes and the important school/college-university transition, and to take appropriate steps.

With the reforms as laid out by ALCAB, DfE, (ALCAB, 2014a,b), (DfE, 2014a,b), Ofqual's publication of Subject Level Conditions and Requirements (Ofqual, 2016a,b), Guidance (Ofqual, 2016c,d), and further exemplification (Ofqual, 2015), and publication of Specifications and the all-important SAMs (either accredited, or with accreditation pending) ahead of Autumn 2017, HEIs have much more time ahead of Autumn 2019 to plan than teachers in schools and colleges have had to prepare for first teaching in Autumn 2017.

4. Details of the changes: the 'big eight'

The changes that have taken place and which colleagues will need to become familiar with are eight-fold, and we outline the key points and links to further information and resources for each. We conclude, in Section 6, with links to the relevant specifications from the AOs, and the all-important SAMs, all of which will give those responsible for curricula in departments in HEIs a clearer idea of what students are expected to achieve, and through that a perspective on the changes in the knowledge, skills, and understanding new entrants will have from 2019 onwards, including an appreciation of the fresh approach they will have to learning mathematics, and the way in which they have been taught mathematics. (Note that the first three below, (i)-(iii) apply to all reformed AS and A levels and not just those in mathematics and further mathematics.)

i. **Linearity**. All A levels, including Mathematics and Further Mathematics, are now 'linear', which means that the final grade achieved is based solely on a series of papers (typically 3 two-hour papers for Mathematics, and potentially slightly more, shorter papers for Further Mathematics depending on which AO's specification is being followed) taken at the end of the course of study, typically two years in length, and assessing the *whole* of the Suject Content.

ii. **Decoupled**. AS qualifications continue, but the results from an AS qualification will not contribute to the corresponding A level qualification, be that Mathematics or Further Mathematics.

iii. **Synoptic**. With both AS and A levels being linear, each qualification is intended to be synoptic, with any examination question being able to draw from across the whole of the content in the relevant qualification. While there are no options within Mathematics (see iv.), Further Mathematics retains some optionality, but again within each qualification, regardless of the various strands that are combined to form the qualification, the intention is that this is also synoptic in the sense that examination questions will be able to draw from across the whole of the relevant content.

iv. **100% prescribed content, including mechanics and statistics**. ALCAB's primary aim was: '… to provide modern A levels that: contain the necessary material; will be interesting to learn and teach; will serve HE and employment'.

    That last part is crucial and acknowledged the very wide group of end-users of both AS and A level Mathematics and Further Mathematics. While wishing to ensure that the new AS/A levels would be 'fit for purpose' and serve the undergraduate mathematics community well, particularly at leading universities, ALCAB also wanted to ensure that the current increase in numbers taking AS and A level Mathematics was sustained, and that the success of AS and A level Further Mathematics, much of which is down to the success of the FMSP, continues.

ALCAB sought to develop AS/A levels whose main aims are to:

a. build from GCSE;
b. introduce calculus and its applications;
c. emphasise how mathematical ideas are interconnected;
d. show how mathematics can be applied to model situations mathematically;
e. make sense of data;
f. understand the physical world;
g. solve problems in a variety of contexts;
h. and, above all, prepare students for further study and employment in a wide range of disciplines.

To achieve these aims ALCAB recommended that the content of the single Mathematics AS and A level be fully prescribed. This would ensure all students had covered the same content, from which they could build upon with some degree of reliability in HE or employment.

It was also essential to go into considerable detail with the recommended content, which was lacking in the previous AS/A levels. The content would also need to ensure that co-teaching of pure mathematics between the single A level in Mathematics and AS level Further Mathematics can be achieved.

As AS/A level Mathematics emphasises how mathematical ideas are interconnected and how mathematics can be applied: to model situations mathematically using algebra and other representations; to help make sense of data; to understand the physical world; and to solve problems in a variety of contexts, including social sciences and business. It prepares students for further study and employment in a wide range of disciplines involving the use of mathematics. As such the reformed A levels now include compulsory, prescribed content in the two core applications of the pure mathematics at this level: mechanics and statistics.

v. **Use of data in statistics**. A significant change in the reformed AS and A levels is in the requirement that, as part of their study of statistics, students:

a. become familiar with one or more specific large data set(s) in advance of the final assessment (these data must be real and sufficiently rich to enable the concepts and skills of data presentation and interpretation in the specification to be explored);
b. use technology such as spreadsheets or specialist statistical packages to explore the data set(s);
c. interpret real data presented in summary or graphical form;
d. use data to investigate questions arising in real contexts.

Specifications should require students to:

e. explore the data set(s), and associated contexts, during their course of study to enable them to perform tasks that assume familiarity with the contexts, the main features of the data and the ways in which technology can help explore the data;
f. demonstrate the ability to analyse a subset or features of the data using a calculator with standard statistical functions.

The intention is that, rather than students focussing on performing routine calculations to determine summary statistics, they should use technology to do this and then focus on the understanding and interpretation of these statistics.

vi. **Overarching themes**. This is a fundamental part of the reforms. Having stated the aims of the new AS/A levels:

a. understand mathematics and mathematical processes in a way that promotes confidence, fosters enjoyment and provides a strong foundation for progress to further study;
b. extend their range of mathematical skills and techniques;
c. understand coherence and progression in mathematics and how different areas of mathematics are connected;
d. apply mathematics in other fields of study and be aware of the relevance of mathematics to the world of work and to situations in society in general;
e. use their mathematical knowledge to make logical and reasoned decisions in solving problems both within pure mathematics and in a variety of contexts, and communicate the mathematical rationale for these decisions clearly;
f. reason logically and recognise incorrect reasoning;
g. generalise mathematically;
h. construct mathematical proofs;
i. use their mathematical skills and techniques to solve challenging problems which require them to decide on the solution strategy;
j. recognise when mathematics can be used to analyse and solve a problem in context;
k. represent situations mathematically and understand the relationship between problems in context and mathematical models that may be applied to solve them;
l. draw diagrams and sketch graphs to help explore mathematical situations and interpret solutions;
m. make deductions and inferences and draw conclusions by using mathematical reasoning;
n. interpret solutions and communicate their interpretation effectively in the context of the problem;
o. read and comprehend mathematical arguments, including justifications of methods and formulae, and communicate their understanding;
p. read and comprehend articles concerning applications of mathematics and communicate their understanding;
q. use technology such as calculators and computers effectively and recognise when such use may be inappropriate;
r. take increasing responsibility for their own learning and the evaluation of their own mathematical development;

ALCAB wanted to make clear that merely providing a list of detailed Subject Content was not enough to bring about the changes required to make AS/A levels reflect the nature of mathematics at this level. This was achieved by introducing three ‘Overarching Themes’ (OTs) into the Subject Content that encapsulate the knowledge and skills that students should be required to demonstrate:

I. OT1 Mathematical argument, language and proof
II. OT2 Mathematical problem solving
III. OT3 Mathematical modelling

and that these must be applied, along with associated mathematical thinking and understanding, across the whole of the detailed content.

It is important for undergraduate programmes in mathematics, but also for the wider group of end-users in HE, that mathematics is not just seen as a collection of techniques to be mastered. The OTs should permeate the teaching of the subject content, so that ‘the whole is more than the sum of its parts’. ALCAB had no remit over assessment, but it hoped that assessments would reflect these wider intentions. Indeed, it identified a number of issues with the current A levels which are intrinsically connected to assessment (ALCAB, 2014a).

vii. **Use of technology**. The use of technology, in particular mathematical and statistical graphing tools and spreadsheets, must permeate the study of AS and A level Mathematics.

viii. **Further mathematics**. The structure of Further Mathematics has changed significantly. The aims remain broadly the same as Mathematics, and is designed for students with an enthusiasm for mathematics, many of whom will go on to degrees in mathematics, engineering, the sciences and economics. The qualification is both deeper and broader than A level Mathematics.

AS and A level Further Mathematics build from GCSE level and AS and A level Mathematics. As well as building on algebra and calculus introduced in A level Mathematics, the A level Further Mathematics core content introduces complex numbers and matrices, fundamental mathematical ideas with wide applications in mathematics, engineering, physical sciences and computing.

The non-core content includes different options that can enable students to specialise in areas of mathematics that are particularly relevant to their interests and future aspirations. A level Further Mathematics prepares students for further study and employment in highly mathematical disciplines that require knowledge and understanding of sophisticated mathematical ideas and techniques.

AS Further Mathematics, which can be co-taught with A level Further Mathematics as a separate qualification and which can be taught alongside AS or A level Mathematics, is a very useful qualification in its own right. It broadens and reinforces the content of AS and A level Mathematics, introduces complex numbers and matrices, and gives students the opportunity to extend their knowledge in applied mathematics and logical reasoning. This breadth and depth of study is very valuable for supporting the transition to degree level work and employment in mathematical disciplines.

In terms of structure and content:

a. A level Further Mathematics has a prescribed core which must comprise approximately 50% of its content. For the remaining 50% of the content, different options are available. The content of these options is not prescribed and will be defined within the different AOs’ Specifications; these options could build from the applied content in A level Mathematics, they could introduce new applications, or they could extend further the core content, or they could involve some combination of these. Any optional content must be at the same level of demand as the prescribed core.
b. In any AS Further Mathematics specification, at least one route must be available to allow the qualification to be taught alongside AS Mathematics: the content of the

components that make up this route may either be new, or may build on the content of AS Mathematics, but must not significantly overlap with or depend upon other A level Mathematics content.

c. At least 30% (approximately) of the content of any AS Further Mathematics specification must be taken from the prescribed core content of A level Further Mathematics, and 20% (approximately) of the overall content of AS Further Mathematics is prescribed. AOs must select other content from the remainder of the core content of A level Further Mathematics to be in their AS Further Mathematics specifications; this should represent a minimum of 10% (approximately) of the AS Further Mathematics content.

It is worth noting that Professor Sir Adrian Smith's groundbreaking report published in 2004: 'Making Mathematics Count' (Smith, 2004), identified many of the failings of the curriculum for mathematics at the time, much of which have prevailed until the current reforms:

> 'It is the consensus view that far too much time is devoted to examinations and preparing for examinations – "teaching to the test" – and that this is at the expense of the understanding of the subject itself. Many identify the problem as the splitting of the subject matter of A-level mathematics into six separately examined modules. This is seen as having the effect of splintering the unity and connectedness of the mathematics to be learned at this level. It is felt that this fragmented presentation makes it virtually impossible to set genuinely thought-provoking examination questions that assess the full range of mathematical skills. It is also felt that the style of short examination papers results in a race against the clock that adversely affects weaker candidates.'

The Smith report also includes reference to the Advanced Extension Award (AEA) (Pearson, 2017a) which is an *additional* entry qualification (to A level) favoured by some universities. It is both interesting and reassuring to note that the AEA has some of the key features that ALCAB placed great importance on, in terms of the Overarching Themes and other aspects highlighted in (i)-(viii) above. (AEA is by no means the only *additional* qualification (to A level) favoured, or indeed required, for entry to some of the 'top universities', including many in the Russell Group. See also MEI, 2016.) As stated in the Smith report, AEA:

> 'aims to enable students to:
>
> - demonstrate their depth of mathematical understanding;
> - draw connections from across the subject;
> - engage with proof to a much greater extent than is required in A-level mathematics.
>
> Questions on the AEA paper are much longer and less structured than those in the modular papers. They require a greater level of understanding than for GCE A-level as well as the ability to think critically at a higher level. The AEA is not expected to require the teaching of additional content, but requires exposure to deeper forms of reasoning and rigour, and a less compartmentalised approach to problem solving. Students are awarded additional marks for their ability to develop creative, and perhaps unexpected, solutions to problems.'

It is therefore to be hoped (and expected) that some of the positive attributes of the AEA will feature in the reformed A levels in Mathematics (and Further Mathematics).

5. Assessment

The next crucial step to releasing the full potential of these recommendations is then placed in the hands of the AOs offering the qualifications, together with the regulator, Ofqual, who, in turn, is responsible for scrutinising the Specifications and Sample Assessment Materials provided by the AOs to establish whether or not these meet the relevant criteria. Ofqual make this assessment through extensive use of a highly-experienced External Panel of Subject Experts.

But what are these criteria?

In the main these comprise Assessment Objectives, since, ultimately, students will sit examinations and these must reflect the purpose of the qualification and, in this case, ALCAB's intentions as set out above, including the all-important Overarching Themes. Ofqual achieve this, in part, by setting out Subject Level Conditions and Requirements (Ofqual, 2016a,b), together with Guidance (Ofqual, 2016c,d).

The Assessment Objectives for AS/A level Mathematics, against which marking schemes for assessments are reviewed and which these must be compliant with, are shown in the table below.

	Objective	Weighting (AS level)	Weighting (A level)
AO1	**Use and apply standard techniques** Learners should be able to: • select and correctly carry out routine procedures; and • accurately recall facts, terminology and definitions	60%	50%
AO2	**Reason, interpret and communicate mathematically** Learners should be able to: • construct rigorous mathematical arguments (including proofs); • make deductions and inferences; • assess the validity of mathematical arguments; • explain their reasoning; and • use mathematical language and notation correctly. *Where questions/tasks targeting this assessment objective will also credit Learners for the ability to 'use and apply standard techniques' (AO1) and/or to 'solve problems within mathematics and in other contexts' (AO3) an appropriate proportion of the marks for the question/task must be attributed to the corresponding assessment objective(s).*	20%	25%
AO3	**Solve problems within mathematics and in other contexts** Learners should be able to: • translate problems in mathematical and non-mathematical contexts into mathematical processes; • interpret solutions to problems in their original context, and, where appropriate, evaluate their accuracy and limitations; • translate situations in context into mathematical models; • use mathematical models; and • evaluate the outcomes of modelling in context, recognise the limitations of models and, where appropriate, explain how to refine them. *Where questions/tasks targeting this assessment objective will also credit Learners for the ability to 'use and apply standard techniques' (AO1) and/or to 'reason, interpret and communicate mathematically' (AO2) an appropriate proportion of the marks for the question/task must be attributed to the corresponding assessment objective(s).*	20%	25%

Each of the Assessment Objectives are subdivided into Strands, and each Strand is also subdivided into Elements (Ofqual, 2016c,d). It is these, together with further exemplification provided by Ofqual in these regulatory documents, that an AO's Specifications and SAMs must be compliant for the associated qualifications offered by them to receive accreditation, and thence candidates for these to be awarded an AS or A level qualification in Mathematics. Similar Conditions, Requirements, and Guidance, with variation to allow for the optional component, apply to Further Mathematics (Ofqual, 2016b,d).

Clearly AO2 and AO3 map onto, respectively, OT1 and OT2, and additional exemplification of the Guidelines for the Awarding Organisations, particularly in respect of mathematical problem solving, modelling, and the use of large datasets, all of which are integral to these reforms, have also been provided (Ofqual, 2015), each of which can be summarised as follows:

I. **Mathematical problem solving**

> 'Mathematical problem solving is a key feature of GCSE and AS/A level Mathematics. It is clear from the subject-content documents for these qualifications that mathematical problem solving is not just for the highest-achieving candidates: it is a core part of mathematics that can and should be accessible to the full range of candidates. For AS and A level Mathematics and Further Mathematics, mathematical problem solving is described in OT2. These OTs are a set of descriptions intended to inform and shape the teaching and learning of AS and A level Mathematics and Further Mathematics.
>
> One way to explore how best to assess problem solving is to consider the possible attributes of assessment of problem solving tasks. (Problem solving tasks in this context are understood to mean a set of requirements focusing on one problem. These tasks may be broken down into a number of steps or parts (that is, items), but this should not undermine the expectation for AS/A level candidates to demonstrate their ability to solve problems as a coherent process.)
>
> The following list contains examples of some of these attributes. These would be expected to be present in tasks that focus primarily on the assessment of problem solving, but may also arise in questions designed primarily to assess other aspects of the detailed subject content and that contain a problem solving element. It is not necessary for every problem solving task to exhibit all of the following attributes, although at least one attribute should apply for a task to be regarded as problem solving:
>
> A. Tasks have little or no scaffolding: there is little guidance given to the candidate beyond a start point and a finish point. Questions do not explicitly state the mathematical process(es) required for the solution.
> B. Tasks provide for multiple representations, such as the use of a sketch or a diagram as well as calculations.
> C. The information is not given in mathematical form or in mathematical language; or there is a need for the results to be interpreted or methods evaluated, for example, in a real-world context.
> D. Tasks have a variety of techniques that could be used.
> E. The solution requires understanding of the processes involved rather than just application of the techniques.
> F. The task requires two or more mathematical processes or may require different parts of mathematics to be brought together to reach a solution.'

(Note that: not all of these attributes would be required within a single task to establish it as problem solving; neither does the presence of one or more attributes within a task automatically imply problem solving is taking place.)

II. **Modelling**

‘Mathematical modelling is covered comprehensively in the Subject Content in a variety of different contexts. For the purposes of assessment, modelling is currently included in the same Assessment Objective (AO3) as problem solving. As with problem solving, modelling is encapsulated within the Overarching Themes in the Mathematics and Further Mathematics Subject Content documents, as OT3.

The content requires candidates to construct their own models, as well as to use known and given models and assumptions, reflecting on the potential impact of their modelling assumptions.’

III. **Large data sets in statistics**

‘The subject content for A level Mathematics requires candidates to be familiar with one or more specific large data sets, to use technology to explore the data set(s) and associated contexts, to interpret real data presented in summary or graphical form, and to use data to investigate questions arising in real contexts. This requirement reflects a desire to change the way in which statistics is taught, and this has implications for assessment.’

6. Specifications and Sample Assessment Materials (SAMs)

The first part of reforms is therefore almost complete, with accredited Specifications and SAMs available for first teaching in September 2017. The next part will be soon underway with students embarking on the new AS/A levels, and so we conclude with links to the relevant Specifications and SAMs for the different providers of qualifications that are on offer, with an AS/A level Mathematics and AS/A level Further Mathematics available for each.

The ones that are on offer are:

- AQA - AS and A level Mathematics and Further Mathematics (AQA, 2017) and available via http://www.aqa.org.uk/subjects/mathematics/as-and-a-level.
- Pearson - AS and A level Mathematics and Further Mathematics (Pearson, 2017b) and available via https://qualifications.pearson.com/en/qualifications/edexcel-a-levels/mathematics-2017.html.
- OCR A - AS and A level Mathematics and Further Mathematics (OCR, 2017a) and available via http://www.ocr.org.uk/qualifications/by-subject/mathematics/as-a-level-maths-from-2017/.
- OCR B (MEI) - AS and A level Mathematics and Further Mathematics (OCR, 2017b) and available via http://www.ocr.org.uk/qualifications/by-subject/mathematics/as-a-level-maths-from-2017.

We encourage teaching staff, and programme and module leads, to review all of these. They will give those responsible for curricula in departments in HEIs a clearer idea of what students are expected to achieve, and through that a perspective on changes: in the knowledge, skills and understanding new entrants will have from 2019 onwards; their approach to learning mathematics; and the way in which they have been taught mathematics.

7. Summary

In concusion, we believe it important that all colleagues should be made aware of the substantive changes here: mechanics and statistics becoming compulsory; the change in emphasis for statistics, including the use of large data sets, particularly to inform teaching; the importance of the Overarching Themes: Mathematical argument, language and proof; Mathematical problem solving; and most importantly the removal of scaffolding in some questions compared with current practice.

8. References

ACME (2017a). Advisory Committee on Mathematics Education. Available via https://royalsociety.org/about-us/committees/advisory-committee-on-mathematics-education-acme/ and http://www.acme-uk.org/home [last accessed 15 April 2017].

ACME (2017b). Advisory Committee on Mathematics Education. Available via http://www.acme-uk.org/about-acme [last accessed 15 April 2017].

ACME (2017c). Advisory Committee on Mathematics Education. Available via http://www.acme-uk.org/about-acme/the-committee/profiles [last accessed 15 April 2017].

ALCAB (2013). A Level Content Advisory Board. Available via https://alcab.org.uk/our-people/subject-panels/ [last accessed 15 April 2017].

ALCAB (2014a). A Level Content Advisory Board Report. Available via https://alevelcontent.files.wordpress.com/2014/07/alcab-report-on-mathematics-and-further-mathematics-july-2014.pdf [last accessed 15 April 2017].

ALCAB (2014b). A Level Content Advisory Board Report: Executive Summary. Available via https://alevelcontent.files.wordpress.com/2014/07/alcab-mathematics-report-executive-summary.pdf [last accessed 15 April 2017].

ALCAB (2014c). A Level Content Advisory Board: Letter from Richard Craster to the Department for Education. Available via https://alevelcontent.files.wordpress.com/2014/07/letter-from-alcab-mathematics-and-further-mathematics-chair-to-dfe-8-july-2014.pdf [last accessed 15 April 2017].

ALCAB (2014d). A Level Content Advisory Board: Letter from Nigel Thrift to the Secretary of State. Available via https://alevelcontent.files.wordpress.com/2015/01/letter-nigel-thrift-to-secretary-of-state-31-october-2014.pdf [last accessed 15 April 2017].

ALCAB (2014e). A Level Content Advisory Board: Letter from Nigel Thrift to the Secretary of State. Available via https://alevelcontent.files.wordpress.com/2014/12/letter-with-annexes-from-alcab-chair-to-secretary-of-state-18-november-2014.pdf [last accessed 15 April 2017].

ALCAB (2014f). A Level Content Advisory Board: Letter from the Secretary of State to Nigel Thrift. Available via https://alevelcontent.files.wordpress.com/2015/01/letter-s-of-s-to-alcab-chair-25-november-2014.pdf [last accessed 15 April 2017].

AQA (2017). AQA - AS and A level Mathematics and Further Mathematics. Available via http://www.aqa.org.uk/subjects/mathematics/as-and-a-level [last accessed 15 April 2017].

DfE (2010). Schools White Paper: The importance of teaching. Available via https://www.gov.uk/government/publications/the-importance-of-teaching-the-schools-white-paper-2010 [last accessed 15 April 2017].

DfE (2013a). Letter from Secretary of State to Glenys Stacey. Available via https://www.gov.uk/government/uploads/system/uploads/attachment_data/file/278146/SoS_January_2013_ofqual_letter_alevels_v2.pdf [last accessed 15 April 2017].

DfE (2013b). Letter from the Secretary of State to Glenys Stacey. Available via https://www.gov.uk/government/uploads/system/uploads/attachment_data/file/278171/SoS_to_glenys_stacey_14-03.pdf [last accessed 15 April 2017].

DfE (2013c). Letter from Nigel Thrift to the Secretary of State. Available via http://webarchive.nationalarchives.gov.uk/20130123124929/http://media.education.gov.uk/assets/files/pdf/p/letter%20to%20sos%2031%20may.pdf [last accessed 15 April 2017].

DfE (2013d). Letter from the Secretary of State to Nigel Thrift. Available via https://www.gov.uk/government/uploads/system/uploads/attachment_data/file/278210/sos_reply_to_thrift.pdf [last accessed 15 April 2017].

DfE (2013e). Letter from Glenys Stacey to the Secretary of State. Available via http://webarchive.nationalarchives.gov.uk/20141031163546/http://ofqual.gov.uk/documents/letter-prilto-secretary-of-state-on-gcse-and-a-level-reform-september-2013/ [last accessed 21 April 2017].

DfE (2014a). GCE AS and A level subject content for mathematics. Available via https://www.gov.uk/government/publications/gce-as-and-a-level-mathematics [last accessed 15 April 2017].

DfE (2014b). GCE AS and A level subject content for further mathematics. Available via https://www.gov.uk/government/publications/gce-as-and-a-level-further-mathematics [last accessed 15 April 2017].

FMSP (2017a). Further Maths Support Programme. Available via http://furthermaths.org.uk/ [last accessed 15 April 2017].

FMSP (2017b). Further Maths Support Programme: CPD events. Available via http://furthermaths.org.uk/cpd_events [last accessed 15 April 2017].

Maths Hubs (2017). Maths Hubs. Available via http://www.mathshubs.org.uk/ [last accessed 15 April 2017].

MEI (2016). Understanding the UK Mathematics Curriculum Pre-Higher Education. MEI and sigma. Available via http://mei.org.uk/files/pdf/pre-university-maths-guide-2016.pdf [last accessed 15 April 2017].

OCR (2017a).OCR A - AS and A level Mathematics and Further Mathematics. Available via http://www.ocr.org.uk/qualifications/by-subject/mathematics/as-a-level-maths-from-2017/ [last accessed 15 April 2017].

OCR (2017b). OCR B (MEI) – AS and A level Mathematics and Further Mathematics. Available via http://www.ocr.org.uk/qualifications/by-subject/mathematics/as-a-level-maths-from-2017/ [last accessed 15 April 2017].

Ofqual (2015). A Level Mathematics Working Group Report on Mathematical Problem Solving, Modelling and the Use of Large Data Sets in Statistics in AS/A Level Mathematics and Further Mathematics. Available via https://www.gov.uk/government/uploads/system/uploads/attachment_data/file/481857/a-level-mathematics-working-group-report.pdf [last accessed 15 April 2017].

Ofqual (2016a). GCE subject-level conditions and requirements for mathematics. Available via https://www.gov.uk/government/publications/gce-subject-level-conditions-and-requirements-for-mathematics [last accessed 15 April 2017].

Ofqual (2016b). GCE subject-level conditions and requirements for further mathematics. Available via https://www.gov.uk/government/publications/gce-subject-level-conditions-and-requirements-for-further-mathematics [last accessed 15 April 2017].

Ofqual (2016c). GCE subject-level guidance for mathematics. Available via https://www.gov.uk/government/publications/gce-subject-level-guidance-for-mathematics [last accessed 15 April 2017].

Ofqual (2016d). GCE subject-level guidance for further mathematics. Available via https://www.gov.uk/government/publications/gce-subject-level-guidance-for-further-mathematics [last accessed 15 April 2017].

Pearson (2017a). Edexcel – Advanced Extension Award (AEA). Available via https://qualifications.pearson.com/en/qualifications/edexcel-a-levels/advanced-extension-award-mathematics-2008.html [last accessed 15 April 2017].

Pearson (2017b). Edexcel – AS and A level Mathematics and Further Mathematics. Available via https://qualifications.pearson.com/en/qualifications/edexcel-a-levels/mathematics-2017.html [last accessed 15 April 2017].

RS (2014). Vision for mathematics and science education, The Royal Society. Available via https://royalsociety.org/topics-policy/projects/vision/ [last accessed 15 April 2017].

Smith, A. F. M. (2004). Making Mathematics Count: The report of Professor Adrian Smith's Inquiry into Post-14 Mathematics Education. Available via http://www.mathsinquiry.org.uk/report/MathsInquiryFinalReport.pdf [last accessed 15 April 2017].

Smith, M. E. (2014). Results of the consultation on revised A level subject content. Available via https://www.gov.uk/government/uploads/system/uploads/attachment_data/file/301060/Post-consultation_report_on_A_levels.pdf [last accessed 15 April 2017].

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[i] The author is in the unique position of having been the only person involved in all aspects of this process through being a member of the: A Level Content Advisory Board (ALCAB); Ofqual External Subject Expert Panel on Conditions and Requirements, and Guidance; Ofqual Working Group on Mathematical Problem Solving, Modelling and the Use of Large Data Sets in Statistics; Ofqual External Subject Expert Panel (as the Overarching Reviewer) for the accreditation of all AS and A Levels in Mathematics and Further Mathematics.

CASE STUDY

Addressing maths anxiety and engaging students with maths within the curriculum

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Abstract

Maths anxiety is a situation-specific anxiety condition which is particularly prevalent in mature students. Previous negative learning experiences with maths condition the brain into maths avoidance behaviours which impacts on students' choices, their self-efficacy, and their curriculum progression. Otherwise-capable students find themselves unable to study effectively and put-off studying until the last minute. This paper discusses some of the strategies implemented as part of an ongoing collaborative project between the Department for Lifelong Learning (DLL), the Mathematics and Statistics Help centre (MASH), and the Specialist Learning Difference (SpLD) service at the University of Sheffield, the aim of which is to reduce anxiety and engage students to the learning of university mathematics.

Keywords: maths anxiety, embedding, formative tests, flipped learning.

1. Introduction

Maths anxiety can be described as "an emotion that blocks a person's reasoning ability when confronted with a mathematical situation" (Spicer 2004), and is negatively related to mathematics achievement (Woodard 2004). The brain of the maths-anxious student shows an activation of emotional areas when confronted with a mathematics problem, the strength of which is correlated with the degree of maths anxiety. This wastes working memory and attentional resources to address the anxiety, instead of allocating them to the maths problem (Young et al., 2012). This means that whilst the student is in a state of anxiety, their brain is unable to process maths effectively. Lyons and Beilock (2012) also showed that the anticipation of doing maths actually activates brain regions associated with pain, although the act of actually doing maths does not, which demonstrates that maths anxiety is a conditioned anticipatory fear of maths. It is therefore not surprising that the main behavioral symptom of maths anxiety is maths avoidance (Woodard, 2004; Ashcraft and Krausse, 2007). For students with maths anxiety, the brain views maths as a visceral threat to be avoided at all costs. This usually means opting-out of maths as soon as possible, and where this is not possible, avoidant behavioural strategies are employed, such as not studying regularly, and leaving study until the last minute, are common (Jackson, 2008; Woodard, 2004). Given this approach, students with high levels of maths anxiety are 'at-risk' of failing, which impacts on progression and student retention rates.

The Department for Lifelong Learning (DLL) at the University of Sheffield provides Foundation courses, primarily for mature students, in which all students have to study a core maths module. Due to the background and disrupted schooling of the DLL cohort, they often have high levels of anxiety, especially surrounding subjects perceived as 'difficult' in school, such as maths. Staff at DLL and the Maths and Statistics Help centre (MASH), which offers 1:1 help with mathematics and statistics education, both observed the impact of anxiety in students over several years. It was decided that a curriculum-based collaborative approach, encouraging attendance at MASH, was necessary to address the issues associated with maths anxiety. The project team have been trialing and evaluating changes to the maths curriculum to improve student confidence over two academic years and this paper evaluates the strategies implemented in 2015/16 and discusses the changes to the curriculum in 2016/17.

2. Strategies

The first step for addressing maths anxiety is to raise student awareness of what maths anxiety is and how their negative beliefs are affecting them (Uusimaki and Kidman, 2004; Martinez and Martinez, 1996). Once students understand the issues, they are able to start addressing their anxiety and engaging with other teaching strategies.

Students with maths anxiety tend to find the classroom stressful as they are afraid of looking stupid in front of their peers, so a simple strategy is to release the lecture notes and resources early enough for anxious students to read through them in advance. DLL students are now given access to the core learning materials in advance of the lessons, and use contact time to focus on processing and implementing the knowledge.This is known as a flipped learning model. In a study by Charles-Ogan and Williams (2015), those using the flipped learning approach had a significantly higher increase in maths test scores when compared to a group receiving traditional teaching.

Having only one exam at the end of a course (high-stakes testing) exacerbates maths anxiety, whilst untimed, unassessed, repeatable (low-stakes) tests actually reduce maths and test anxiety as well as boosting confidence (Simzar et al 2015). Feedback is known to help reduce the negative impact of maths anxiety on academic achievement; unfortunately, feedback tends to be limited in most courses (Núñez-Peña et al., 2015). Formative test scores are a form of feedback as students are being reassured of their knowledge and alerted to topics in which they are weaker (Friedman, 1987), which is especially important for a sequential curriculum such as maths. Anonymous online tests are particularly useful as students can check their understanding without peers knowing their score, and after the initial design, can be used on further cohorts of students. Multiple retrials of formative tests allows students to rewrite past feelings of failure by observing their performance improvements (Juhler, Rech, From, and Brogan, 1998). This is most effective when the question format is kept constant, but the numbers in the question are randomly generated.

3. Methods

3.1. Awareness and background information

In order to raise awareness of maths anxiety and begin to address students' issues, a maths anxiety workshop was designed by MASH in collaboration with the SpLD service at the University of Sheffield. The workshop is delivered to DLL students in the first maths lesson each year. The session is embedded into the timetabled schedule, rather than being optional and voluntary, as maths-anxious students are unlikely to attend events with "maths" in the title. The workshop uses an active, reflective approach and aims to increase awareness, dispel common maths myths, and encourage engagement with the suggested strategies (Marshall et al, in press). Visual representations are used to summarise recent neuroscience research so that students can understand how anxiety affects the brain and why maths avoidance is common. The usage of brain images makes psychological arguments more convincing (McCabe and Castel, 2008), which strengthens the notion that the students' anxieties are due to neurological factors, rather than a notion that they "can't do maths".

In Autumn 2015, all students taking the Foundation Maths class were asked to fill in a short survey prior to the workshop and to give feedback straight after to evaluate the workshop. One of the questions asked the students to rate their levels of maths anxiety before the session on a 5-point Likert scale ranging from "None At All" to "Very High". As the whole class attended the session, it was important to distinguish between the groups when assessing the effectiveness of the strategies. In addition, they were encouraged to fill in the "Attitudes to Maths" survey which went out to all students at the University of Sheffield prior to the workshop. This questionnaire included the 23-item UK Maths Anxiety Rating scale (MAS-UK) which was constructed by Hunt et al. (2011), and is a reduced version of the Maths Anxiety Rating Scale (MARS) originally developed by Richardson and Suinn (1972). The results of the whole survey were published in the 2016 MSOR conference proceedings (Marshall et al, 2016).

3.2. Formative assessment

Weekly self-check tests were created in the university virtual learning environment (VLE) which allowed information on who used the tests to be collected. By allowing parameter randomisation in the online questions each time a new test was started, students could revise topics where they had made mistakes and try again without repeating the same question. Feedback can be added to these online tests, along with links to online materials that give students an alternative to the lecture notes. If students got a question wrong, an example of how to approach the question was given in the feedback and, where available, a link to a further resource on the topic was given. In February 2016, students were asked to fill in another survey to investigate the usefulness of the self-check tests for reducing anxiety and increasing confidence, along with other questions on their strategies for studying effectively. Students were also asked to fill in the MAS-UK scale again to assess changes from baseline.

3.3. Classroom strategies

Given that past negative learning experiences are the main source of maths anxiety, creating a positive learning environment is vital. After further research on building maths confidence, the module leader adopted a flipped learning model. This approach promotes independent thinking and flexible learning by allowing access to lecture material and additional online resources such as videos and web pages, in addition to tutor-created slides, well in advance of the lesson. In keeping with the success of the VLE-based tests from the previous year, new and more extensive online tests with parameter randomisation were created using Numbas (numbas.org.uk), a free open-source e-assessment tool by Newcastle University (Foster, Perfect and Youd, 2012). These tests were then embedded within the university VLE and made available to students alongside the learning resources.

During pre-lesson study time, students could submit questions on Padlet (padlet.com), an online notice board that would then be answered during a face-to-face interactive lecture. Padlet was chosen as it allows students to ask questions anonymously, avoiding the risk of humiliation in class, and thus reducing their maths anxiety. The interactive lecture was designed to clarify specific points for the whole cohort, and also to gauge what the students had learned from the flipped resources. In order to do this in an anonymous but engaging way, multiple-choice quizzes were given using the Plickers (plickers.com) platform. Students vote for an answer to each question by holding up a unique paper-based QR code, a random-looking pattern corresponding to one of the four options. Plickers quizzes give students immediate feedback on their understanding whilst enabling them to view the distribution of answers across the multiple choices by the whole class. It is also a useful tool to enable the lecturer to identify common areas of misunderstanding, and address them immediately.

In addition to the flipped learning resources and the interactive lectures, students also attended a follow-up tutorial where they were able to practice non-assessed exam-style questions in a tutor- and peer-supported environment. The module leader asked for feedback about these new strategies implemented in the 2016/17 academic year so far. This feedback was given via a brief questionnaire asking how enjoyable and useful students found the various resources; there was also an open-text box for students to give additional feedback if desired. The new strategies will be more formally evaluated in the future using the exam results, the end-of-module feedback, and observed changes in engagement within the classroom.

4. Results

4.1. Comparison of maths attitudes results

The results of the maths attitudes survey for the rest of the University of Sheffield were compared with the responses for the DLL cohort to look for differences. A Mann-Whitney U-test showed a significant difference in maths anxiety scores ($U=16221.5$, $p=0.015$) between the DLL cohort and the rest of the participants. The median anxiety score for DLL was 22.5 compared to 16.0 in the general university population. Previous research on this dataset showed that absence of A-level Mathematics was the strongest predictor of high maths anxiety (Marshall et al., 2016). As this is a Foundation course, only 10% of DLL students filling in the survey had A-level Mathematics, and 39% had either a GCSE Maths grade below a C or alternative numeracy qualifications instead. For students who had taken GCSE/O-level maths, 95% of the general university population achieved a GCSE grade A*-C on their first attempt, compared to only 61% of the DLL students. A Mann-Whitney test on the full survey data showed a significant difference ($U=6112$, $p=0.001$) in maths anxiety scores for those who did and did not achieve a GCSE grade A*-C on the first attempt. The median anxiety score for those achieving A*-C on the first attempt was 15 compared to 29 for those who didn't.

4.2. Awareness workshop feedback

91% of the 57 students filling in the awareness workshop feedback agreed or strongly agreed that the session was interesting and 87% agreed that the session was useful. 44% of the 57 respondents filling in the feedback survey classified themselves as having moderate to very high levels of maths anxiety. The self-reported scores were strongly related ($\rho=0.67$) to the official MAS-UK measures for those who had taken part in both surveys. 78% of those classified as moderate to very high levels of anxiety felt less anxious as a result of the session and 59% felt more confident about their maths ability. A lot of positive feedback about the session and the impact it had on students self-confidence was also given to departmental staff during the term, shown by some of the qualitative student feedback:

"I first started the course with a huge amount of maths anxiety. Since the first MASH maths anxiety session, I have been able to control my anxiety which has helped me understand maths better. I feel much more confident to give maths questions a go and have the ability to figure out where I went wrong."

"I am more confident in dealing with maths since starting in September. It is reassuring knowing support is there if and when needed."

4.3. Formative assessment

Overall, 80% of students took at least one of the nine initial online self-check tests created for the 2015/16 cohort and 51% completed at least three different tests. Those in the high anxiety group completed the most different tests on average although the difference was not significant (median=6.5). Only 20 students filled in the questionnaire about the online tests, and of these, only 14 had initially classified themselves as having moderate to high anxiety. As part of the survey, students were given 5-point scales to rate helpfulness from 'unhelpful' to 'helped a great deal'. They were asked about different aspects of the test and how helpful the aspect was regarding reducing anxiety, increasing confidence and increasing understanding. Figure 1 summarises the percentages who found the tests in general, and the option for retesting, helpful.

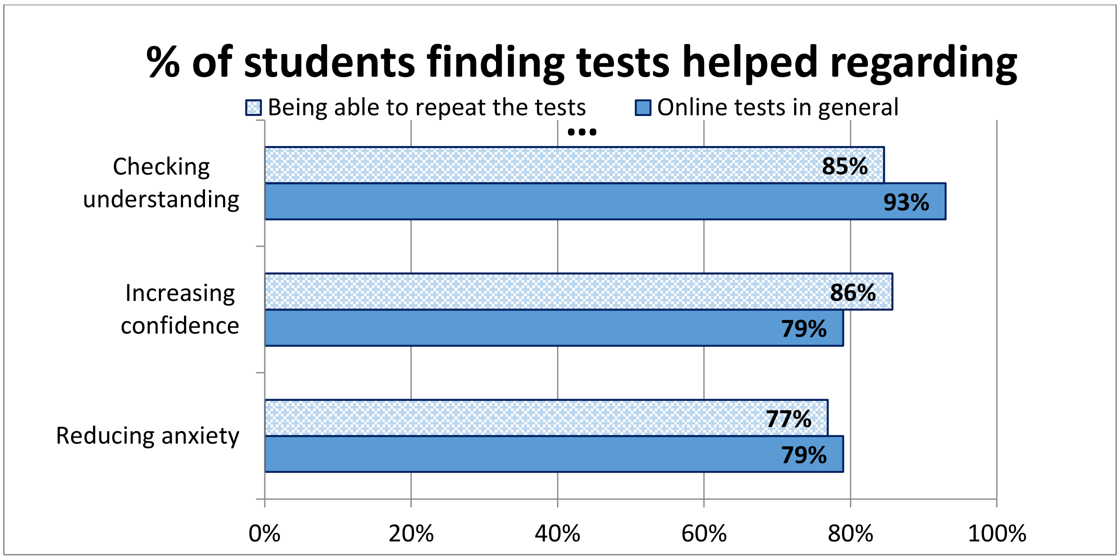

Figure 1: Survey feedback summary of percentage of students with moderate to high anxiety who said aspects of the online tests helped at least a bit.

It is clear from the results in Figure 1 that most students found the online tests helpful. This was also reinforced by student comments:

"*Using the tests to check my understanding helped out massively when I came to finding out the material I knew and didn't."*

Due to these results, further formative, repeatable online tests were created and implemented in the second year of the study, using the Numbas platform.

4.4. Classroom strategies

27 students from the DLL 2016/17 cohort filled in a brief, informal questionnaire asking how enjoyable and useful they found each aspect of the course, based on 5-point scales from 'I hated it' to 'I enjoyed it a lot', and from 'useless' to 'very useful', respectively. The questionnaire asked about the 'aspects of the course', i.e. the new resources designed and curated for the academic year 2016/17. Figure 2 summarises these results by showing the percentages of students who found each aspect enjoyable or useful, by choosing either of the two positive options in each 5-point scale.

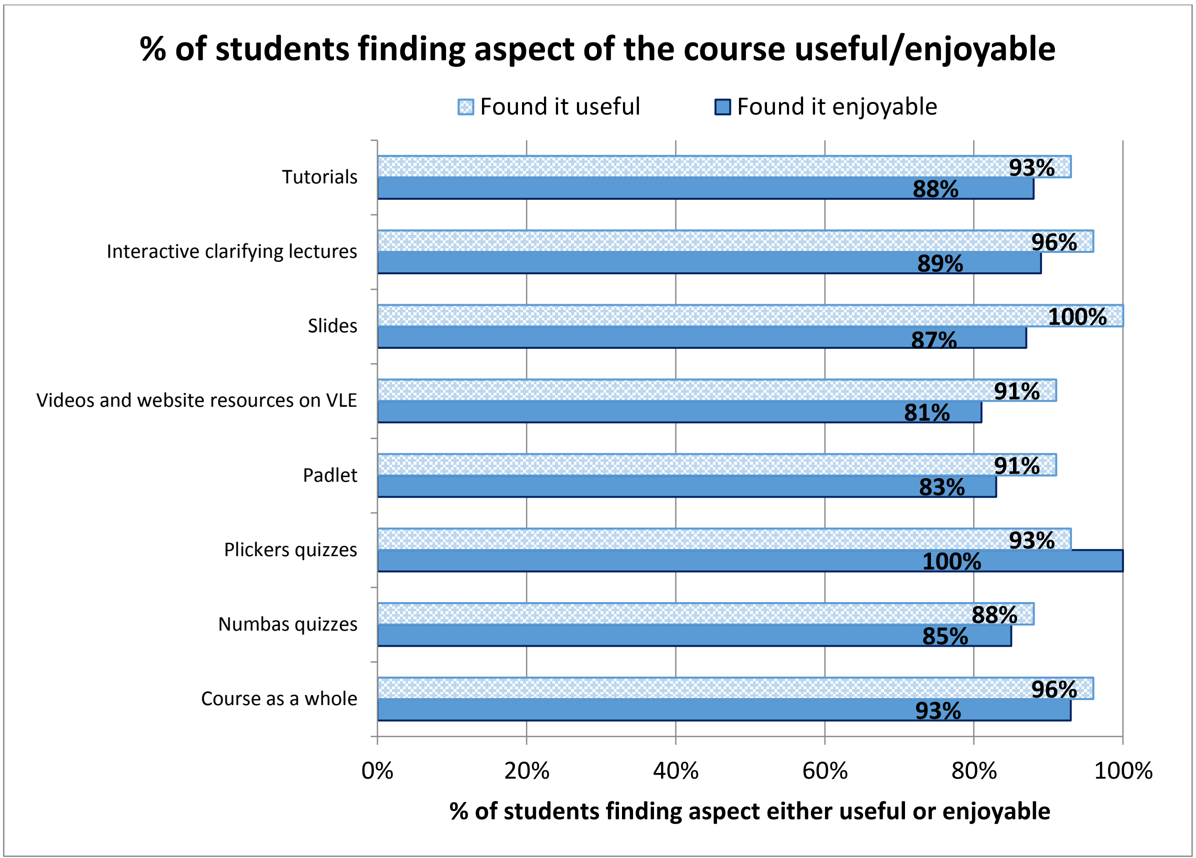

Figure 2: Summary of informal questionnaire feedback showing the percentages of students who found the various aspects of the course enjoyable or useful.

Based on the feedback, it is clear that students found the various resources available both useful and enjoyable. It is interesting to note that, with the exception of the Plickers quizzes, students recognised that the resources were very useful even if they didn't find them enjoyable. There was still a high percentage of students who enjoyed each aspect of the course, which has helped to reduce their anxiety levels, as evident from some of the free-text comments:

"The sessions are helpful and fun and have helped to lessen my Maths anxiety substantially. I like the Plickers quizzes for the quick thinking aspect and to cement what we have learned from the weekly materials."

The comments also indicate that the students found the learning environment presented by the tutor to be positive and encouraging.

"...you are taught in an academic manner, but are made to feel as though you can ask any question without feeling silly. Explanations are given clearly. The weekly materials are posted 2 weeks in advance which is helpful to me as it allows me to prioritise my time. The tutorials are helpful in that we can stay to ask questions if we are struggling but if we feel we are okay we can reflect on our answers in our own time."

"*Since attending my Foundation course I have come to realise that maths is actually quite fun, and that if you get the base right, the rest can follow. I've learned to be a lot calmer and approach what it is that I'm doing more positively."*

More detailed feedback on which aspects students found most helpful and which helped reduce their anxieties will be collected and reported later in the semester.

5. Conclusion

Maths anxiety is an issue impacting on many students' mathematical understanding in the Department for Lifelong Learning, and a number of strategies were trialled to address the issue. This paper demonstrates that implementing awareness, behavioral and formative test strategies are useful for addressing maths anxiety within the curriculum. The collaborative approach employed in this project has benefited the staff and students involved, and from the very first week of teaching encouraged students to use the various methods of support to succeed with their course. Whilst the initial preparation of the flipped learning approach and creation of formative online tests is quite time consuming, the benefits to the students are clear. Through the research carried out by the team, we hope to demonstrate to other academics that by embedding approaches that reduce anxiety within the curriculum, and encouraging the use of additional university teaching services, all students can reach their full potential.

6. References

Ashcraft, M.H. and Krause, J.A., 2007. Working memory, math performance, and math anxiety. *Psychon. Bull. Rev.,* 14(2).

Foster, B., Perfect, C., and Youd, A., 2012. A completely client-side approach to e-assessment and e-learning of mathematics and statistics. *International Journal of e-Assessment*, 2(2).

Friedman, H., 1987. Repeat examinations in introductory statistics. *Teaching of Psychology,* 14(1), pp. 20-23.

Hunt, T. E., Clark-Carter, D., and Sheffield, D., 2011. The development and part validation of a UK scale for mathematics anxiety. *Journal of Psychoeducational Assessment,* 29(5), pp.455-466.

Jackson, E., 2008. Mathematics Anxiety in Student Teachers. *Practitioner Research in Higher Education,* 2(1), pp. 36-42.

Juhler, S. M., Rech, J. F., From, S. G., and Brogan, M. M., 1998. The effect of optional retesting on college students' achievement in an individualized algebra course. *Journal of Experimental Education*, 66(2), pp. 125–137.

Lyons, I. M., and Beilock, S. L., 2012. When math hurts: math anxiety predicts pain network activation in anticipation of doing math. *PloS One*, 7(10), https://doi.org/10.1371/journal.pone.0048076.

Marshall, E.M., Wilson, D.A., and Mann, V.E., 2016. Attitudes and anxiousness about maths. In *Brave New World*: *Proceedings of the CETL MSOR Conference 2016*, Loughborough, UK, pp. 66-74.

Marshall, E.M., Wilson, D.A., and Mann, V.E. (in press). Evaluating the effectiveness of maths anxiety awareness workshops. *Journal of Learning Development in Higher Education.*

Martinez, J. G. R. and Martinez, N. C., 1996. Math without fear. Needham Heights, MA: Allyn and Bacon.

McCabe, D.P. and Castel, A.D., 2007. Seeing is believing: The effect of brain images on judgements of scientific reasoning. *Cognition,* 10(7), pp. 343-352.

Núñez-Peña, M. I., Bono, R., and Suárez-Pellicioni, M., 2015. Feedback on students' performance: A possible way of reducing the negative effect of math anxiety in higher education. *International Journal of Educational Research*, 70, pp. 80-87.

Charles-Ogan, G. and Williams, C., 2015. Flipped classroom versus a conventional classroom in the learning of mathematics. *British Journal of Education,* 3(6), pp.71-77.

Uusimaki, L. S. and Kidman, G. C., 2004. Reducing maths-anxiety: Results from an online anxiety survey. *Presented at the AARE Annual Conference, Melbourne, Australia: Australian Association for Research in Education.*

Richardson, F. C., and Suinn, R. M., 1972. The Mathematics Anxiety Rating Scale: Psychometric data. *Journal of Counseling Psychology,* 19(6), pp. 551.

Simzar, R. M., Martinez, M., Rutherford. T., Doina, T., and Conley, A.M., 2015. Raising the stakes: How students' motivation for mathematics associates with high-and low-stakes test achievement. *Learning and individual differences,* 39, pp. 49-63.

Spicer, J., 2004. Resources to combat math anxiety. *Eisenhower National Clearinghouse Focus,* 12(12).

Woodard, T., 2004. The Effects of Math Anxiety on Post-Secondary Developmental Students as Related to Achievement, Gender, and Age. *Inquiry: A Journal of Medical Care Organization, Provision and Financing*, 9(1).

Young, C. B., Wu, S. S., and Menon, V., 2012. The neurodevelopmental basis of math anxiety. *Psychological Science,* 23(5), pp. 492–501.

CASE STUDY

Motivating Mature Students of Mathematics

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Abstract

This paper considers mature students who take a pre-degree Certificate in Science at a university in Ireland. In particular, it focuses on typical challenges that these students face in their mathematics education, and discusses some motivational techniques their lecturer used in an attempt to address student concerns. It situates these techniques within the general theory and considers how to measure their impact.

Keywords: Mature Student, Motivation

1. Introduction

Every year, Maynooth University admits 15-25 students to The Certificate in Science Programme (the Certificate). This one-year course is designed for mature students returning to education. For the purposes of this course, a mature student is classified as anybody over 22 years of age, on the first of January in the year they commence the course. These students typically would not meet the entry requirements for a place on a degree programme and would be recommended to complete the Certificate. Provided the student achieves a specific overall grade in the Certificate, a place would be offered to them on a degree level programme in Maynooth University.

Students begin the course in mid-September with a proficiency examination to gauge their general mathematical ability. It tests basic mathematics such as sets, arithmetic, algebra and trigonometry. The median grade was 35% for the 2014-15 class group, and 33.5% for the following year group.

Early in both years, students communicated to the course lecturer (first author) regarding their previous negative experiences in education. In particular, they focused on mathematics, and their concern that it was a barrier to their progression to their degree of choice. Based on this information, the lecturer, who had taken a Psychology of Education module during his teacher training, decided to draw on this knowledge in an attempt to motivate the group and encourage them to keep the required momentum to pass the module.

This paper considers the three most frequent scenarios related to student motivation which were encountered in these classes. We examine some prominent theories of motivation and discuss how they were implemented in the classroom. We close with an analysis of the quantitative and qualitative data available from both groups in an attempt to determine what effect, if any, the strategies implemented had on student motivation.

2. Literature Review

Motivation may be defined as "…a force that energizes and directs behaviour toward a goal" (Eggen & Kauchak, 1994: p.427) and is often categorised as either extrinsic or intrinsic (Merriam & Bierema, 2013). Extrinsic motivation is "…motivation that occurs from reinforcers, feedback, or rewards that are not inherent in the activity itself" (Good & Brophy, 1995: p.402). The person is often not interested in the task, but rather is concerned with what they can gain by doing it. On the other hand, intrinsic motivation is an individual's own natural tendency to want to seek out solutions to problems. "The "reward" for engaging with the task lies in the pleasure and sense of satisfaction that are inherent in engagement itself" (Anderman & Anderman, 2010: p.30). We should nurture intrinsic drive in our pupils; however this can also be supported by extrinsic influences. We will now examine different approaches to extrinsic and intrinsic motivation and consider how they complement each other.

Behavioural approaches to motivation (extrinsic) are centred on the ideas of positive and negative reinforcement. From an educational perspective, the aim is to encourage a particular behaviour or attitude in students using these reinforcers. The idea is that "All of the infinite variety of human behaviour can be made more or less frequent or probable by the use or non-use of reinforcement, contingent on some response" (Gage & Berliner, 1992: p.231).

There are a number of common methods used in the classroom to bring about behavioural modification. Sometimes we offer simple rewards or punishments/sanctions in order to promote hard work or good behaviour. In particular, praise (Biehler & Snowman, 1997) and constructive feedback can be a powerful tool in encouraging students to work. This recognition can make a pupil feel good about themselves and encourage them to keep working and engaging in class. The quality of feedback that students receive impacts their self-confidence (Capel & Gervis, 2005).

Some researchers argue that the use of reinforcers can negatively impact on the intrinsic motivation to learn (Eggen & Kauchak, 1994) and the student is simply trying to impress a teacher rather than having a desire to solve a problem. However, Jordan et al (2008: p.34) argue that "Behaviourism is not totally antagonistic to other theories of learning; rather it can co-exist with later learning theories that focus on cognition or the social acquisition of meaning."

Social approaches to motivation (extrinsic) are concerned with the impact on motivation that may be obtained from the social interactions of students with their teachers, peers, etc. The teacher should be an individual who is "…warm, understanding, friendly, responsible, systematic, imaginative and enthusiastic…" (Fontana, 1995: p.384). Indeed "A student who identifies with and admires a teacher of a particular subject may work hard partly to please the admired individual and partly to try becoming like that individual" (Biehler & Snowman, 1997: p.400). For this reason, the educator should aim to foster a mutual respect between the students and teacher in order to create an environment where learning can take place.

Peers are another key social motivator in the learning experience. It is natural that an individual finds a social group that shares similar thoughts, opinions and beliefs as themselves. "Students learn together in class, while friends, classmates and study partners learn together outside of college campus" (Lei, 2010: p.156). If the peer group places a high value on academic success then it is likely that the individual will aim to conform to this ethos. These social groups can have a positive effect on an individual as they can be a wonderful source of intellectual as well as emotional support.

The humanistic approach to motivation (intrinsic) “stresses students’ capacity for personal growth, freedom to choose their destiny, and positive qualities” (Santrock, 2009: p.461). One key ingredient of this humanistic approach is the development of self-efficacy. “In general, self-efficacy is a person’s self-constructed judgement about his or her ability to execute certain behaviours or reach certain goals” (Ormrod, 2008: p.356). The psychologist Albert Bandura identified the following four factors that affect self-efficacy:

Mastery Experience:

“Successes build a robust belief in one’s personal efficacy” (Bandura, 1998: p.624). If a pupil has previously succeeded in a similar task or subject area, they are more inclined to approach new material with a degree of enthusiasm and confidence. It is important that challenges posed to students are appropriate for their learning level. If the task is too simple, while the student may receive an immediate increase in self-efficacy, it will not teach them perseverance for more difficult problems or situations.

Vicarious Experiences:

Bandura (1998: p.626) states that “Seeing people similar to oneself succeed by sustained effort raises observers’ beliefs that they too possess the capabilities to master comparable activities to succeed.” For example, if a pupil has reservations about whether or not they can solve a mathematics problem, one might reassure them by illustrating how students in the past have had similar concerns but were in fact successful at completing the task.

Verbal Persuasion:

Bandura (1998: p.626) explains that “People who are persuaded verbally that they possess the capabilities to master given activities are likely to mobilize greater effort and sustain it than if they harbour self-doubts and dwell on personal deficiencies when problems arise.” Praise can boost a student’s self-efficacy and contribute to that internal feeling that they can succeed. Educators must make a conscious effort to explicitly identify, for the sake of the pupil, how they have excelled in a particular area, be it an academic subject or an extra-curricular activity. Goals set by both the educator and the student must be achievable.

Physiological State:

How a human interprets their physiological reactions to situations, determines whether they get a boost to their self-efficacy or not. For example, as part of a project, a student may have to give a class presentation. Some pupils may have feelings of anxiety around facing their classmates. “They interpret their stress reactions and tension as signs of inefficacy” (Bandura, 1998: p.626). Depending on the learner’s level of self-confidence, they may either embrace the challenge or shy away from the task. It is therefore vital that teachers do all in their power to control situations like this to ensure a positive outcome.

The cognitive approach is another intrinsic source of motivation. From this viewpoint, people are seen as having a desire to seek out solutions to problems. They have a natural curiosity and when a topic is personally relevant, they require little incentive to pursue an answer. It is important that the teacher, when planning a lesson, makes the material as relevant as possible for the students. With reference to mature students, Knowles (2012: p.257) argues that we should "…use the existing knowledge experience and motivation of learners to shape the learning experience."

One approach is to use inquiry based learning. "Inquiry-based education is a learner-centered form of teaching and learning that enables students to tailor at least some of their learning experiences to their own interests and curiosity" (Saunders et al, 2012: p.17). The teacher assumes the role of a facilitator rather than a source of information and the class can be divided into groups because

> …the shared responsibility and interaction produce more positive feelings toward tasks and others, generate better intergroup relations and result in better self images for students with histories of poor achievement (Joyce et al, 1997: p.89).

This approach encourages the pupils to ponder higher order questions and not shy away from challenges.

3. Common Scenarios

Each year, common scenarios relating to student motivation occurred on a regular basis. The lecturer observed that members of the class who presented with these scenarios could be loosely categorised into three groups. It should be noted that some students fell into more than one category. We briefly describe these scenarios for the academic years 2014-15 and 2015-16 and we provide detail on the various techniques employed, in an attempt to address the issues raised. These methods were implemented during every lecture (6 per week).

3.1 Students with Mathematical/Educational Baggage

Students in this category (approximately two thirds of the class in both years) were asking questions, working in groups and regularly availing of the optional extra supports available in the Mathematics Support Centre (MSC). The lecturer noticed early on in the module, that these students had negative knee-jerk reactions to new material presented in lectures. Often these occurred before the students had engaged with or even read the material in question. Remarks such as "this is really difficult" or "I'm never going to get this" were frequently heard aloud in class. While these students were not necessarily at risk of failing, the concern was that this mentality would snowball and impact on the atmosphere of learning for all students within the class. In an attempt to counteract this, the lecturer implemented a number of motivational strategies, principally using vicarious and cognitive approaches.

Prior to new material being presented in class, the teacher would remind the group that it is normal for new material to seem difficult when seen for the first time. Vicarious approaches (Bandura, 1998), such as informing the class that previous groups who had taken the same module had similar initial reactions. However, when they engaged with the material, they realised that it was not as difficult as it first appeared. In addition, class tours of the National Science Museum and the University's Russell Library were organised. This library houses a significant collection of old mathematical texts. These tours included brief presentations on the items being displayed, which demonstrated to the students that the study of mathematics and science is a process that takes time and patience.

Cognitive approaches, as outlined by Knowles (2012), were also used by the lecturer in lesson planning. Problems were chosen which appealed to mature students' experiences and natural curiosity and they were encouraged to work in groups to bring them to a conclusion. For example, when working on basic statistics, the class considered current economic issues, the relevant data, and how they were presented in the media.

3.2 Students with External Concerns

Students placed in this category (approximately one quarter of the class in both years) had informed the lecturer, in one-to-one conversations, that they felt under pressure balancing their studies and their personal lives, e.g. many students had part-time jobs, young children etc. Typically these students did not have serious difficulties with the mathematics in the module; they were engaging and actively working, but the lecturer was concerned that they might drop out due to their external concerns.

Students would often confide that they were doing this course in an effort to improve their employment prospects, while others did not have the opportunity to go to college in the past and doing this course, was to some degree, fulfilling a dream of theirs. In either case, these behavioural motivators (Gage & Berliner, 1992) were used to remind them that this short term pain, effort and stress could ultimately lead to them realising their goals. They were reminded of the educational supports available to them and that social motivation (Lei, 2010) should be sought from their classmates as many of them were in similar situations.

Time management issues were also exacerbating the problem for some students. In these circumstances, the lecturer made suggestions to help ease the student’s burden, e.g. scale down on volunteer activities in the local community. Most students took this feedback on board, due in large part to the lecturer’s advice to reflect on the motivators that drove them to return to education. Students indicated that this had helped put their priorities in perspective. The effect of these conversations was almost immediate and the lecturer witnessed a significant improvement in their demeanour.

3.3 Passive Students

These students were attending the majority of their classes and doing their assignments but were considered “passive” by the lecturer (approximately one quarter of the class in both years). They were not asking questions in class, often working on their own and were not attending the MSC. The lecturer was concerned that these students were doing the minimum they thought was necessary to get by, while it was clear to him that they would fail if their behaviour continued.

In these cases, they lecturer spoke to the individuals in private. He voiced his concerns in an effort to gain insight into why they were approaching their studies in this manner. In most cases the student simply did not know how to study or were unaware of the effort that was required to be successful. In these situations, they were advised to join one of the study groups that were developing in the class. Students in these groups not only worked together in class, but also studied together in the MSC and outside of the university (Lei, 2010). In this way, it was hoped that they would come to realise that their previous work rate was not up to par. They were verbally persuaded (Bandura, 1998) by the lecturer that there was time to turn it all around and still pass the course.

4. Results

As both academic years progressed, there were clear indicators that most members of the class appeared, to the lecturer, to be more motivated. For example, the frequency of negative comments on the difficulty of material reduced considerably, classes were rarely missed due to external commitments, and students were observed working together on a regular basis.

Unfortunately, a small number of individuals, mostly in the 'Passive Student' category, did not change their approach, and many of these failed to progress from the Certificate. However, based on these classroom observations alone, it is impossible to state categorically that student progress or non-progress was as a result of the lecturer's various interventions. We now attempt to measure student progress with the available qualitative and quantitative data.

At the end of the two academic years, students filled in an anonymous course evaluation form. There were a total of 18 questions, and 33 students completed the questionnaire. The feedback in general was extremely positive. Two questions are particularly relevant to this paper:

Q: "Having taken this module, do you feel your mathematical knowledge and confidence has improved? In what way?"

Thirty students answered this question, and they all had positive responses. Samples include:

> "Yes, I literally knew nothing about maths coming in, now I'm not as scared of it anymore and have a genuine interest in it."

> "My knowledge and confidence has improved so much that it has had a positive impact on my life outside of college too. It has given me more opportunities for a better, successful future."

Q: "Having completed this course, do you feel prepared for future studies in mathematics or other science based subjects?"

There were 31 responses, all were positive in nature. Samples include:

> "I am more prepared and confident with moving on in a science degree"

> "Yes, I feel this course has set me up for the future."

There was limited quantitative data available for analysis. In 2015, Maynooth University Access Office (MAP) (O'Neill & Fitzsimons, to appear) commissioned a report, which detailed various facts and figures for the Certificate for the years 2002-2012. This report included the completion rates for students (n=153) that registered for the Certificate during those years. The results are reproduced in Figure 1.

When we look at the figures for 2014-15 and 2015-16, we see that completion rates of 73% and 86% in 2014-15 and 2015-16 respectively compare favourably with the completion rate of 64% for the years 2002-2012. Completion rates in the individual subjects for the Certificate are not available.

The MAP (O'Neill & Fitzpatrick, to appear) report also considered when Certificate students that progressed to degree studies in Maynooth University were most likely to drop out. It identified non-progression to second year as most common, with an average drop-out rate of 25% for 2002-2012. At the time of writing, we have data for the 2014-15 students only. We found that of the 14 students that progressed to the 1st year of degree studies, 3 failed to progress, which translates to a marginally better rate of 21%.

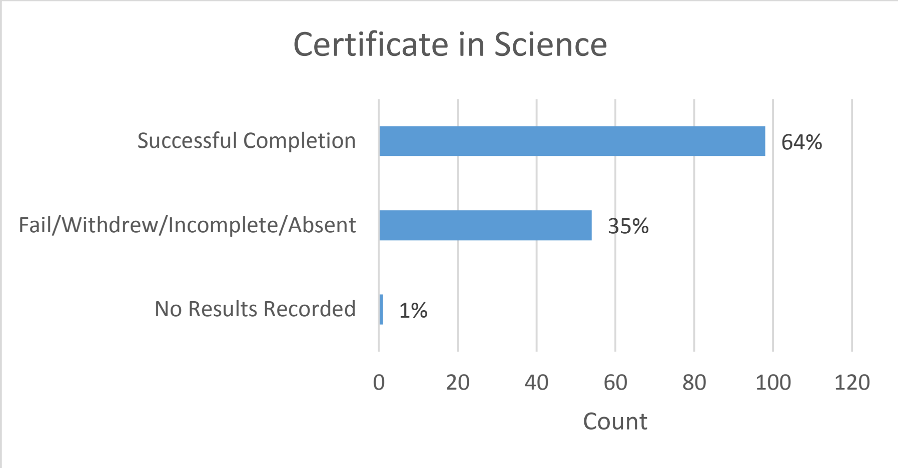

Figure 1. Completion Rates (n=153) for 2002-2012.

5. Conclusion

It is extremely difficult to get a measure of how the implementation of the theories outlined above affected the students. The qualitative data (anecdotal evidence and student feedback) and some of the quantitative data are encouraging. However, the data does not provide sufficient evidence to conclude that the strategies described in this paper were the sole reason for any improvement in student motivation.

Jordan et al. (2008: p.137) reminds us "that adults learn in a different way to children and will use different learning techniques that require different teaching strategies." For this reason, in conjunction with the results of our research, we believe that teachers should be mindful of some theories of motivation, as they can have a positive impact on mature students and help sustain them in their studies.

It is clear that quantitative data alone does not necessarily capture an individual's level of motivation. For example, interviewing students would give additional insight into any changes of attitude or enthusiasm. The data and anecdotal evidence presented suggest that a detailed study with more in-depth analysis would be worthwhile to determine if these results could be repeated on a broader and more sustainable basis.

6. References

Anderman, E. and Anderman, L. (2010). *Classroom Motivation.* New Jersey: Pearson.

Bandura, A. (1998). Health promotion from the perspective of social cognitive theory. *Psychology & Health*, 13(4), pp. 623-649.

Biehler, F. and Snowman, J. (1997) *Psychology Applied to Teaching.* Boston: Houghton Mifflin.

Capel, S. and Gervis, M. (2005). Motivating pupils. In S. Capel, M. Leask and T. Turner, eds. *Learning to teach in the secondary school*. 4th ed. Oxon: Routledge. pp. 120-135

Eggen, P. and Kauchak, D. (1994). *Educational psychology: classroom connections.* 2nd ed. New York: MacMillan College Publishing.

Fontana, D. (1995). *Psychology for Teachers.* 3rd ed. Hampshire: Palgrave.

Gage, N.L. and Berliner, D.C. (1992). *Educational psychology.* 5th ed. Boston: Houghton Mifflin.

Good, T.L. and Brophy, J. (1995). *Contemporary educational psychology.* 5th ed. New York: Longman.

Jordan, A. Carlile, O. and Stack, A. (2008). *Approaches to Learning. A Guide for Teachers.* Berkshire: Open University Press.

Joyce, B. Calhoun, E. and Hopkins, D (1997). *Models of learning – tools for teaching.* Buckingham: Open University Press.

Knowles, M. Holton, E. and Swanson, R. (2012). *The Adult Learner.* 7th ed Oxon: Routledge.

Lei, S.A. (2010). Intrinsic and Extrinsic Motivation: Evaluating Benefits and Drawbacks from College Instructors' Perspectives. *Journal of Instructional Psychology*, 37(2), pp. 153-160.

Merriam, S.B. and Bierema, L.L. (2013). *Adult Learning: Linking Theory and Practice*. Somerset, US: Jossey-Bass.

O'Neill, J. and Fitzsimons, C. Review of Maynooth University Foundation Certificates Programme. *Unpublished manuscript*.

Ormrod, J.E. (2008). *Educational psychology. Developing learners.* 6th ed. New Jersey: Pearson.

Santrock, J.W. (2009). *Educational psychology*. 4th ed. New York: McGraw-Hill.

Saunders-Stewart, K., Gyles, P., and Shore, B. (2012). Student Outcomes in Inquiry Instruction: A Literature-Derived Inventory. *Journal of Advanced Academics*, 23(1), pp. 5-31.

RESEARCH ARTICLE

UniDoodle: A Multi-Platform Smart Device Student Response System – Evaluated in an Engineering Mathematics Classroom

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Abstract

Most of the existing student response systems, such as clickers, have limited input capabilities, typically only offering students a multiple-choice selection. In some instances, students can input a numerical or textual response. However, mathematical equations, diagrams, etc. are all beyond the capabilities of such systems. This paper proposes and presents a novel multi-platform smart device-based student response system, called UniDoodle, that allows for a more generic and flexible input. This system consists of a student application that allows for freeform input through sketching capabilities, a lecturer application that allows easy viewing of multiple sketch-based responses and a cloud-based service for co-ordinating between these two applications. In essence, students can now respond to a question posed by the lecturer using sketches and, hence, mathematical equations, circuit diagrams, graphs, etc. are all possible on the UniDoodle system. In addition, the lecturer can now gain a richer and more useful insight to the students' understanding of the relevant material. This paper also evaluates the UniDoodle system in a large class of first year Engineering Mathematics students. Details of the UniDoodle system, the evaluation process and the feedback obtained are all presented within.

Keywords: student response systems, classroom response systems, formative assessment, technology in the classroom.

1. Introduction

Student response systems (SRSs) are slowly becoming more commonplace in the classroom today. Such systems allow students to respond to in-class questions in real-time, in a fast and efficient manner (Fies and Marshall, 2006). These systems exist under many different guises in the research literature, including clickers (Barber and Njus, 2007), classroom response systems (Roschelle *et al.*, 2004), audience responses systems (Ramesh, 2011, Miller *et al.*, 2003), clicker assessment and feedback technology (Han and Finkelstein, 2013), electronic voting systems (Retkute, 2009) and voting machines (Reay *et al.*, 2005). The literature also clearly outlines the many numerous pedagogical benefits that the use of SRSs entails. These include improved classroom interaction, improved student learning, improved motivation, better retention of material and better attendance (Blasco-Arcas, *et al.*, 2013, Bruff, 2009, Caldwell, 2007, Moredich and Moore, 2007, Rowlett, 2010). In addition, action learning activities and classroom assessment techniques (Angelo and Cross, 1993) are easier implemented through the use of SRSs (Sarason and Banbury, 2004, McLoone *et al.*, 2015).

However, to date, these systems only allow for limited input capabilities, whereby students are required to select an answer from a given set of possible answers, i.e. a multiple-choice selection. In some cases, students can input a numerical or a textual response. Either way, mathematical equations, circuit diagrams, sinusoidal responses, annotations of diagrams, etc. are all beyond the capabilities of existing SRSs.

This lack of freeform input is a significant drawback for the Science, Technology, Engineering and Mathematics (STEM) disciplines where writing equations, drawing circuits and sketching diagrams are important aspects of the student learning experience. For example, consider the solving of an algebraic equation, the designing of a circuit, the sketching of a mathematical function, etc. The list of such examples is endless and it is very important that students of STEM disciplines, in particular, can carry out such fundamental processes and methodology. In order to capture the real-time feedback of the students' grasp of this information it is necessary for a SRS to facilitate freeform input.

Here, we propose the use of a novel multi-platform smart device-based student response system, called UniDoodle, that allows for a more generic and flexible input. Students can now respond to a question posed by the lecturer using sketches and, hence, mathematical equations, circuit diagrams, graphs, etc. are all possible with this system. In addition, the lecturer can access a richer and more useful insight to the students' understanding of the relevant material. This new system operates on any device that supports either the iOS (iPads, iPhones) or Android (smart phones and tablets) operating system.

The UniDoodle system was evaluated by a large class of first year Engineering Mathematics students in the School of Electronic Engineering in Dublin City University (DCU). Both student and lecturer feedback was obtained at the end of the evaluation period. The evaluation process and a summary of the feedback obtained will be outlined in later sections.

2. UniDoodle – a brief overview

UniDoodle is a recently developed, multi-platform, smart device based student response system that provides a freeform-style input using sketch capabilities. It builds on previous work in the area by the lead author (McLoone *et al*., 2015), and currently operates on all devices running either iOS or Android. This system consists of a student application that allows for freeform input, through sketching capabilities, a lecturer application that allows easy viewing and editing of multiple sketch-based responses (see Figure 1) and a Google App Engine cloud-based service for co-ordinating between these two applications.

Overall, the system works as follows – firstly, the teacher poses a question in class (this can be a pre-prepared question in the form of a template or a new on-the-spot question); secondly the students receive this question (in the case of a template question) on their device and can now respond appropriately using the in-built sketch capabilities (equations, diagrams, graphs, and annotations are all possible); thirdly, the teacher receives the student responses in a neat and concise format on their own device (typically a tablet); and lastly the teacher can point out and respond to any obvious errors that students may have made. In addition, the student responses can be viewed via the overhead projector allowing all students to see all the submitted responses. This offers students a level of peer learning as they can now see where other students are making mistakes, if any, and what those mistakes are.

Several new and important features exist in UniDoodle in comparison with the work carried out by McLoone et al. (2015). Firstly, it supports the preparation of questions in advance of class through the use of templates. Teachers can now pre-prepare questions and load them on the cloud-based database. Furthermore, templates can be arranged into various folders (see Figure 2). For example, a teacher could have a different folder of questions for each subject or class they teach. Secondly, diagrams can be created on a PC using any suitable drawing package, saved in any standard image format, and uploaded to the database. This allows for more detailed and precise diagrams to be used as the basis of questions. Finally, UniDoodle includes a filter feature that allows the teacher to remove any undesirable student responses. This quick and easy to use feature means that unwanted images can be deleted before the responses are shown live to the whole class.

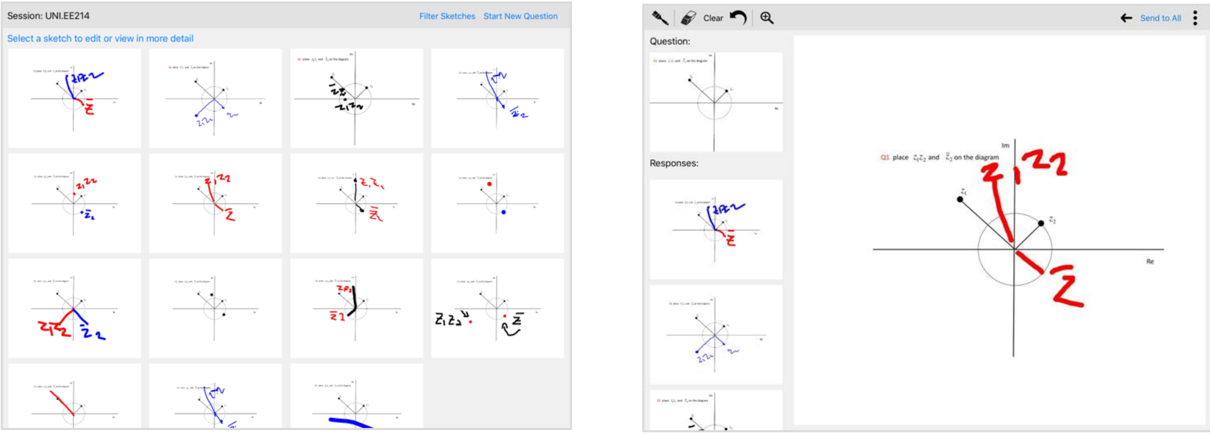

Figure 1. UniDoodle Viewing (left) and Editing (right) Teacher App.

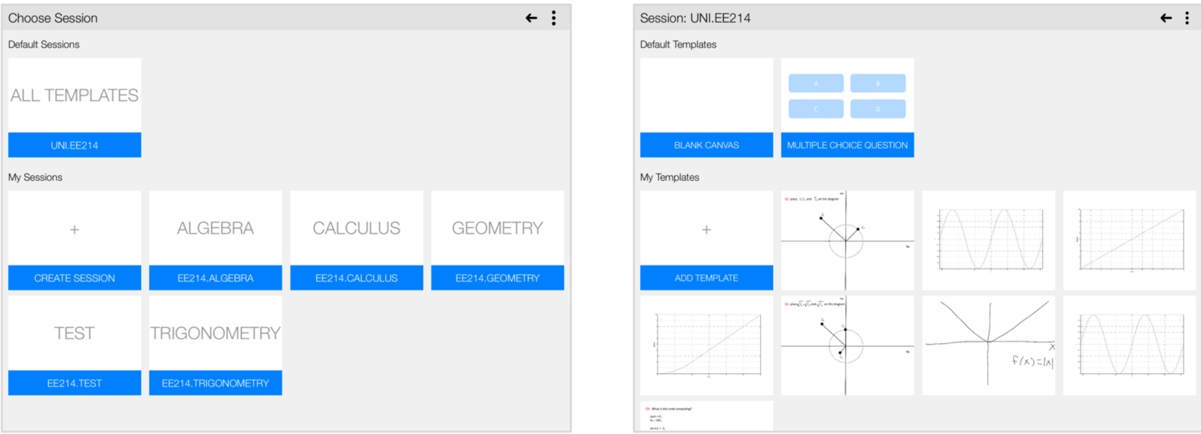

Figure 2. UniDoodle Session Management (left) and Template Development (right).

3. Educational Context

The UniDoodle response system was trialled in two first year modules which are offered by the School of Electronic Engineering in Dublin City University (DCU) to all their first year engineers. These consist of students taking the Common Entry, Electronic and Computer Engineering, Mechatronic Engineering, Mechanical and Manufacturing Engineering and Biomedical Engineering programmes. The modules were EM122 Engineering Mathematics II and EM114 Numerical Problem Solving for Engineers. The former covers traditional topics in calculus, matrices and complex numbers and had 165 registered students, while the latter had 151 registrations and introduces numerical techniques for approximately solving a range of practical problems based on the material covered previously in EM122 and elsewhere.

The UniDoodle system was trialled in 4 distinct 1-hour stand-alone sessions, between February and April 2016, as follows:

> Session 1: Students had to investigate basic concepts in functional analysis. In particular students were asked to sketch, for example, trigonometric functions and investigate the difference between $abs(sin(x))$ and $sin(abs(x))$. They were also asked to sketch the derivative of a given function, based purely on the shape of its graph (no functional description, values or other information was given).
>
> Session 2: Students had to explore more complicated topics in calculus, such as for example, identifying the derivative of $abs(sin(x))$ and $sin(abs(x))$ (two functions that had been explored in Session 1). In addition, applications of calculus were explored with students being presented with sketches of a particle's acceleration against time and asked to produce sketches of its speed and position. Importantly the functional description of acceleration was not given. This means that the student could not simply integrate to find the desired solutions and instead had to consider how the shape of the curve yields the required information.
>
> Session 3: Students had to explore topics from complex numbers, namely the polar form of a complex number and its relationship with the number's position on the complex plane. Students were challenged to include on the diagram various powers or square roots of the given complex number. Again specific information about the arithmetic value of the complex number was not provided. The unit circle was provided for reference.
>
> Session 4: This was a pre-exam session where the multiple choice questions from the previous year's exam paper was reviewed. Students primarily used the multiple choice facility on UniDoodle to submit their answers and engaged in peer-learning to review the submissions and decide, as a group, on the correct answers.

By way of example, the question shown in Figure 3 was used during Session 3. This question was designed to develop the students' knowledge of the use of the polar form of complex numbers, and the ease with which this form allows certain complex arithmetic operations to be carried out. The lecturer had noted how poorly students in previous years tended to perform on related questions in the annual module examination, and felt that UniDoodle would present an opportunity to address misconceptions in advance.

The objective of the question presented was to demonstrate that the square root operation can be estimated very easily by considering the polar form of the complex number. Students are deliberately not given any specific information about the numerical values of the three numbers. This deliberately forces students out of their comfort zone when working with complex numbers, and requires them to make estimations based on an appreciation of the meaning of the argument and modulus of a complex number, rather than blindly performing arithmetical calculations. Demonstrating conceptual understanding in this manner would be difficult without freeform input capabilities, such as those offered by UniDoodle.

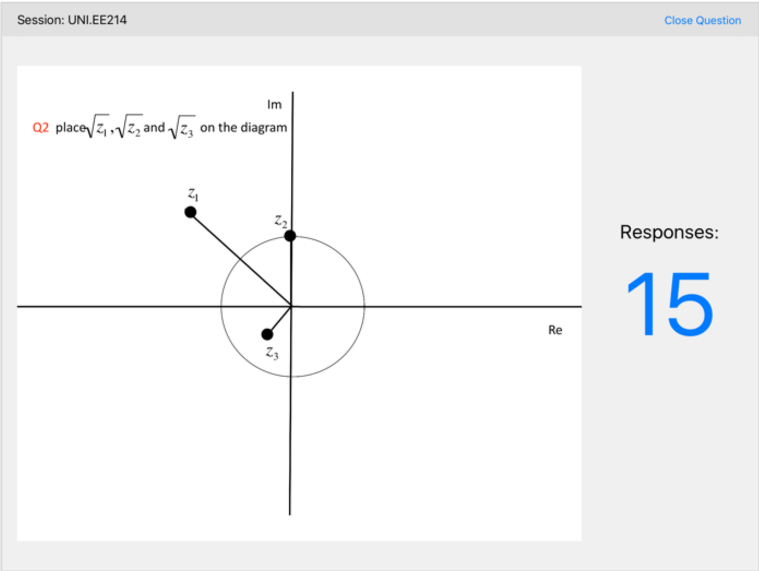

Figure 3. Sample question used in study.

4. Evaluation Method

At the end of the trial period, students were asked to complete detailed survey forms regarding their views on a range of aspects relating to the UniDoodle response system, including (i) usability, (ii) learning using the system and (iii) engagement in the classroom. In addition, the lecturer of the two modules was asked for his personal thoughts and opinion on the use of the UniDoodle system. It is worth noting that the lecturer had used the previous system and this allowed him to express relevant views on some of the new features of the UniDoodle system, as outlined in the previous section.

In total, 98 student survey forms were completed and returned at the end of the evaluation period. A selection of the student feedback is presented in Table 1 and Figure 4. Table 1 gives the average and standard deviation of the responses while Figure 4 shows a percentage breakdown of the responses per question.

5. Results and Discussion

In terms of engagement when using UniDoodle in class, students were generally positive about using the system, expressing that it provided an effective means of interacting with the lecturer. The feedback showed that the majority of students felt that UniDoodle helped them to be active in class, and over 90% of respondents felt that using the system was fun. As expected, most of the students (almost 93%) noted that the anonymity of responses meant that they were more likely to respond to questions. This supports the sentiment echoed in much of the relevant research literature that inidicated anonymity would illicit a higher, and more honest, response from the class (Caldwell, 2007, Rowlett, 2010).

In using UniDoodle, students felt that both they themselves and the lecturer had experienced benefits regarding learning outcomes. The majority expressed that the use of the app allowed the lecturer to identify problem areas, and felt that feedback provided by the lecturer after completing a question helped to improve their understanding of concepts covered. They also stated that

UniDoodle allowed them to measure their own performance and understanding, raised their awareness of what they did not know, and helped them to focus on what they should be learning.

The majority of students were also positive about the usability of the app, finding it easy to use and intuitive to navigate. Most students found it useful to be able to draw sketches using UniDoodle, although a few noted, via additional comments, that some of the drawing aspects could be improved. These included providing a keyboard input for text input, a thinner pen size option for neater sketching and the inclusion of a library of basic geometry shapes for convenience.

Table 1. Average and Standard Deviation of Student feedback regarding UniDoodle (1 to 5 represents strongly disagree, disagree, not sure, agree and strongly agree respectively).

Statement	Average Rating (1–5)	Std. dev.
The use of UniDoodle helped me to be active in class.	4.0	0.9
The fact that my answers were anonymous encouraged me to submit my responses in class.	4.5	0.7
UniDoodle makes me think more about the course material during my lectures.	3.5	1.1
I found it useful to be able to draw sketches with UniDoodle.	4.0	0.8
I found this method of interaction between students and lecturer effective.	4.1	0.8
I would recommend that the lecturer continue to use UniDoodle.	3.7	1.1
The use of UniDoodle allows lecturers to identify problem areas.	3.8	0.9
UniDoodle allows me to better understand key concepts	3.6	0.8
The feedback provided by the lecturer after completing a UniDoodle question helped me focus on what I should be learning in the course.	3.9	0.8
It was easy to use UniDoodle.	4.4	0.8
I rarely had to seek help to use UniDoodle.	4.2	0.9

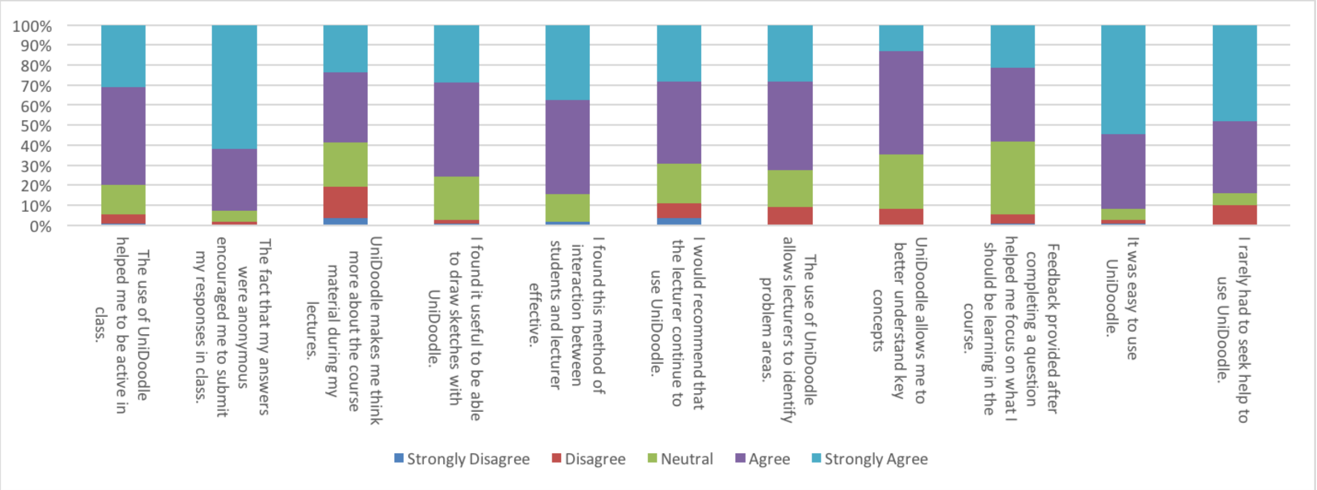

Figure 4. Detailed breakdown of student feedback regarding UniDoodle.

When asked about the disadvantages of using UniDoodle, the most common sentiment was that having their phone out in class presented students with the opportunity to become distracted by other apps, with 54.5% agreeing that they had been distracted. However, when asked to weigh the advantages and the disadvantages overall, for the use of UniDoodle in their module, the vast majority of students felt that UniDoodle was advantageous or very advantageous, with less than 10% disagreeing with this sentiment. In addition, almost 40% of respondents felt that using the app increased their confidence in their ability to complete the module successfully, while 45% were unsure. The reader is referred to Rowlett (2010) for an interesting discussion on the debatable merits of using response systems in a classroom environment, particularly in the context of improving learning outcomes. It should be noted, however, that this discussion predates the UniDoodle system and, so, it will be interesting to observe if UniDoodle can indeed improve learning outcomes. Such a study is beyond the scope of this paper.

The lecturer, who had used the previous version of the SRS, reported that UniDoodle constituted a great improvement. In particular, he found that the extension of the system to iPhones and iPads meant that coverage within the classroom was now effectively 100%. He also noted that since the assessment was not summative, there was no issue with any students who did not have a device (or had forgotten one) as they could pair up with colleagues who did.

The lecturer also felt that the ability to filter responses was a great innovation. The lecturer started each session with a blank template and an invitation to the students to draw a picture of whatever they felt like. Ostensibly this was framed as an invitation to re-familiarise themselves with the basic functionality of the system, while in reality it was intended to quickly address and temper the temptation for students to abuse the system by submitting inappropriate images. The filter facility meant that such submissions could be removed without drawing attention to them and, in practice, students who were intent on disruption quickly realised that there would be no opportunity to do so.

The lecturer found that the template functionality offered a fantastic off-line way of producing good quality questions. However, noting that he occasionally asked on-the-spot questions in the classroom, he added that useful future innovations may include the ability to render simple geometrical figures, text and mathematical content in real-time. Interestingly, this echoed some of the comments given by the students themselves in relation to improving the sketch capabilities on UniDoodle.

As for its use in class, the lecturer found that the students greatly enjoyed using UniDoodle. “The shift from passive observers in a lecture to active participants is something that they really respond to.” In addition, he felt that the anonymous nature of the interaction is vital, noting that it was “astounding that one can ask the simplest question in class to a resounding silence, while a question posed on UniDoodle can receive dozens of responses – many of which can be wrong, but at least have been volunteered”.

Overall, the lecturer observed that the adoption of the UniDoodle system is very worthwhile, but that it was not a trivial undertaking. Devising exercises, which make appropriate use of the system, and test students’ conceptual understanding of the problem, rather than their ability to merely solve a specific numerical example, requires time and practice.

Furthermore, given that he had to quickly examine up to 100 submissions for each question, he found that he had to carefully think about the types of answers that he was likely to receive, as well as common mistakes that were likely to present themselves. This would then allow him to react quickly in the classroom and identify submissions that were worthy of feedback.

6. Acknowledgements

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7. References

Angelo, T. A. and Cross, K. P., 1993. *Classroom Assessment Techniques: A Handbook for College Teachers*. Jossey-Bass, San Francisco, CA.

Barber, M. and Njus, D., 2007. Clicker evolution: seeking intelligent design. *The American Society for Cell Biology*, 6, pp.1-20.

Blasco-Arcas, L., Buil, I., Hernández-Ortega, B. and Sese, F.J., 2013. Using clickers in class. The role of interactivity, active collaborative learning and engagement in learning performance. *Computers and Education*, 62(1), pp.102-110.

Bruff, D., 2009. *Teaching with Classroom Response Systems: Creating Active Learning Environments*. Jossey-Bass, San Francisco, CA.

Caldwell, J.E. 2007. Clickers in the large classroom: current research and best-practice tips. *Life Sciences Education*, 6, pp.9-20.

Fies, C. and Marshall, J., 2006. Classroom response systems: a review of the literature. *Journal of Science Education and Technology*, 15(1), pp.101-109.

Han, J.H. and Finkelstein, A., 2013. Understanding the effects of professors' pedagogical development with Clicker Assessment and Feedback technologies and the impact on students' engagement and learning in higher education. *Computers and Education*, 65, pp.64-76.

McLoone, S., Villing, R. and O'Keeffe, S., 2015. Using mobile touch devices to provide classroom assessment techniques. *Int. Journal of Mobile Human Computer Interaction*, 7(4), pp.1-15.

Miller, R.G., Ashar, B.H. and Getz, K.J., 2003. Evaluation of an audience response system for the continuing education of health professionals. *Journal of Continuing Education in the Health Professions*, 23, pp.109-115.

Moredich, C. and Moore, E., 2007. Engaging student through the use of classroom response systems. *Nurse Educator*, 32(3), pp.113-116.

Ramesh, N., 2011. Use of technology to enhance teaching and learning in mathematics and statistics. *MSOR Connections*, 11(1), pp. 34-36.

Reay, N.W., Bao, L., Pengfei, L., Warnakulasooriya R. and Baugh G., 2005. Towards an effective use of voting machines in physics lectures. *American Journal of Physics*, 73, pp.554-558.

Retkute, R., 2009. Exploring technology-based continuous assessment via electronic voting systems in mathematics and statistics. *MSOR Connections*, 9(1), pp. 24-28.

Roschelle, J., Penuel, W.R. and Abrahamson, A.L., 2004. Classroom response and communications systems; research review and theory. *In Annual Meeting of the American Educational Research Association*, San Diego, CA, pp.1-8.

Rowlett, P., 2010. Ask the Audience (yes, all of them). *MSOR Connections*, 10(1), pp. 3-5.

Sarason, Y. and Banbury, C., 2004. Active learning facilitated by using a game-show format or who doesn't want to be a millionaire? *Journal of Management Education*, 28, pp.509-519.

RESOURCE REVIEW

Knitting Statistics Notes: Creating Teaching Materials with Data Analysis Code and Results Embedded

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Abstract

R is a widely used, free, open source software package for data analysis and graphics. Along with core functionality, there are many additional packages to enable specialist usage. One of these packages ‘knitr' allows the integration of R and LaTeX code. This means that code, output and narrative are all included in one source document. Output is generated automatically from the code. If any changes are made to the data, all output is updated automatically. From a teaching perspective, this is particularly useful if you want to create notes with various graphs, statistics etc and you also want to provide the code that shows the student how to do this analysis in R. It is possible to hide either the code or the output. For example, if you are making introduction to statistics worksheets for non-statistics students, you might not want to include R code. If you are including exercises for students, you might not want to include the output and instead have them answer some questions. You can also set these options globally then by changing one option, create two versions of the document - one with solutions, one without, just from one source document.

Keywords: R, knitr, teaching notes, embedded code, individualised assessments.

1. Background

Literate programming is a programming paradigm conceived by Donald Knuth (Knuth 1984), originally applied to writing software. When developing software, large amounts of documentation are produced, including information for the development team, for maintenance engineers, and for end users, in addition to code comments. The idea of literate programming is to keep the source code for software together with its documentation. Rather than writing instructions to a computer, and then writing documentation to explain the code to a human, literate programming suggests writing a single document that is both the code and its natural explanation and presentation. This is achieved by writing a narrative that interweaves segments of code with text which explains their purpose and operation.

Here we apply the same idea to writing documents such as teaching materials. The narrative is kept together with all code used to produce the output in the document. This means any user (including your future self!) can easily see what data was used, what analysis was done and how (though there are options to hide various elements in the final document if required). This has obvious links to reproducible research - the authors' methods are clearly given along with the output.

2. R

This article is about the knitr package (Xie 2016), which applies literate programming to R (R Core Team 2015). R is a programming environment for data analysis and graphics. While it is a programming environment and is not ‘point-and-click', all but advanced functionality can be achieved using simple commands, so it is relatively easy for a beginner to get started. There is also a big, active internet community meaning there are many free resources and forums available to provide help and guidance.

R is a widely used, free, open source software package and is multi-platform (i.e. it can be used on Windows, Mac or Linux). This means there is very little restriction to its usage in terms of either cost or availability - anyone with access to an internet connection can begin using it right away. There are also no recurring costs such as annual licence renewal fees. Updates are simply downloaded when convenient to the user - you do not find the software stops working on an arbitrary date when the licence runs out.

‘Free' really does mean free - there is no subscription, advertising, hidden spyware or malware. Open source means that anyone with the technical know-how can see exactly what the code does so there is complete transparency of its contents and functionality. Users are able to confirm for themselves exactly what a function does and to suggest improvements to the code.

The standard R installation has core functionality, which is sufficient for a beginner and for common or basic analysis. For more advanced functionality, such as advanced graphics, or specific types of statistical analysis, such as spatial statistics or missing data analysis, R has thousands of specialist packages available to download as required, making it very flexible. Users are also able to create their own packages, which after submission to the R development team for checking, can be made available to all users.

R also includes many datasets as standard. A list of available datasets can be seen by typing ‘data()' into the R console. These are freely available for use and are often used in examples in the R documentation to illustrate how to use a function.

R and its additional packages are available to download from the Comprehensive R Archive Network (CRAN, (R Core Team (2015)). The CRAN website also provides manuals and introductory guides.

The standard installation of R has a basic interface, but there are several additional graphical interfaces which can be more user friendly. Those available as freeware include Tinn-R, RKWard and RStudio. RStudio is particularly helpful for knitr and is discussed in more detail in Section 3. These require R to be installed first, then a separate install for your chosen interface, in much the same way as you would install LaTeX on your system, then your chosen tex editor. Knitr can then be installed using the ‘install.packages()’ function in R.

3. RStudio

In addition to being freeware, RStudio (RStudio, 2016) is multi-platform (runs on Windows, Mac, Linux) and open source. Use of RStudio for working with R generally, or the knitr package specifically, is optional but has several benefits including:

- multiple windows displaying various aspects (discussed below)
- syntax highlighting
- version control
- LaTeX integration

The RStudio interface consists of several windows (see Figure 1):

- Bottom left: console window (also called command window). Here you can type simple commands after the '>' prompt and R will then execute your command.
- Top left: editor window (also called script window). Collections of commands (scripts) can be edited and saved. If you don't see this window initially, you can open it with File -> New -> R script. Typing a command in the editor window will not run it. If you want to run a line from the script window (or the whole script), you can click Run or press CTRL+ENTER to send it to the command window.
- Top right: workspace / history window. In the workspace window you can see which data and values R has in its memory. The history window shows what commands have been typed before.
- Bottom right: files / plots / packages / help window. Here you can open files, view plots, install and load packages or use the help function. You can change the size of the windows by dragging the grey bars between the windows. All the plots you generate will be kept in the plots window and can be exported as pdf, jpg etc.

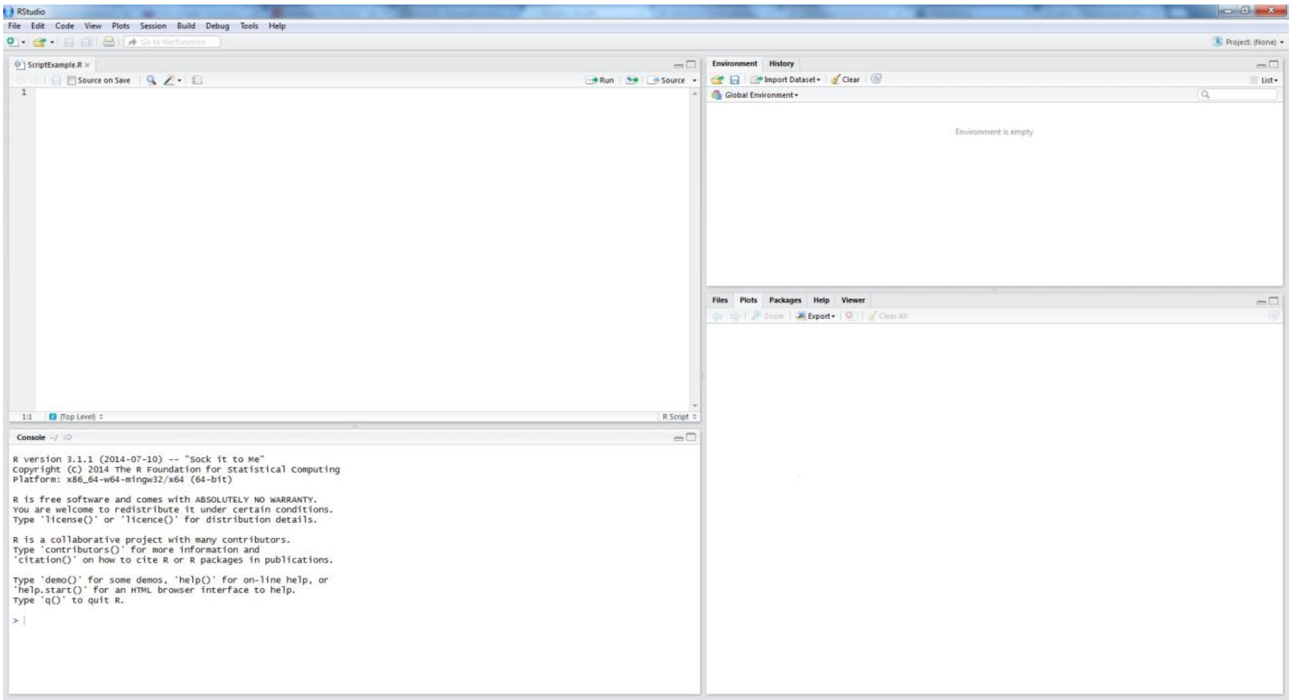


Figure 1. RStudio start up.

4. The knitr Package

The knitr package in R allows the integration of R and LaTeX, employing full LaTeX functionality. This means that R code can be embedded within the LaTeX code, so code, output and narrative are all included within the same document. From a teaching perspective, this is particularly useful if you want to create notes with various graphs, statistics etc and you also want to provide the code that shows the student how to do this analysis in R. If you want to create individualised tasks for students for coursework purposes, you can use random number generation in your R code so each student has different data. You then create the document multiple times, one for each student. Each copy of the document can be created by a single button click (see Section 5.3 for details). All plots, statistics etc in the document would be created automatically using the individualised data. It is also possible to automate the creation of multiple, individualised copies by using simple script, so if you have 100 students, you do not have to create 100 copies yourself. Using a specified random seed means that while data is created/altered randomly, you would be able to recreate it at a later date and see what results a particular student should have.

5. How To

5.1. An Example

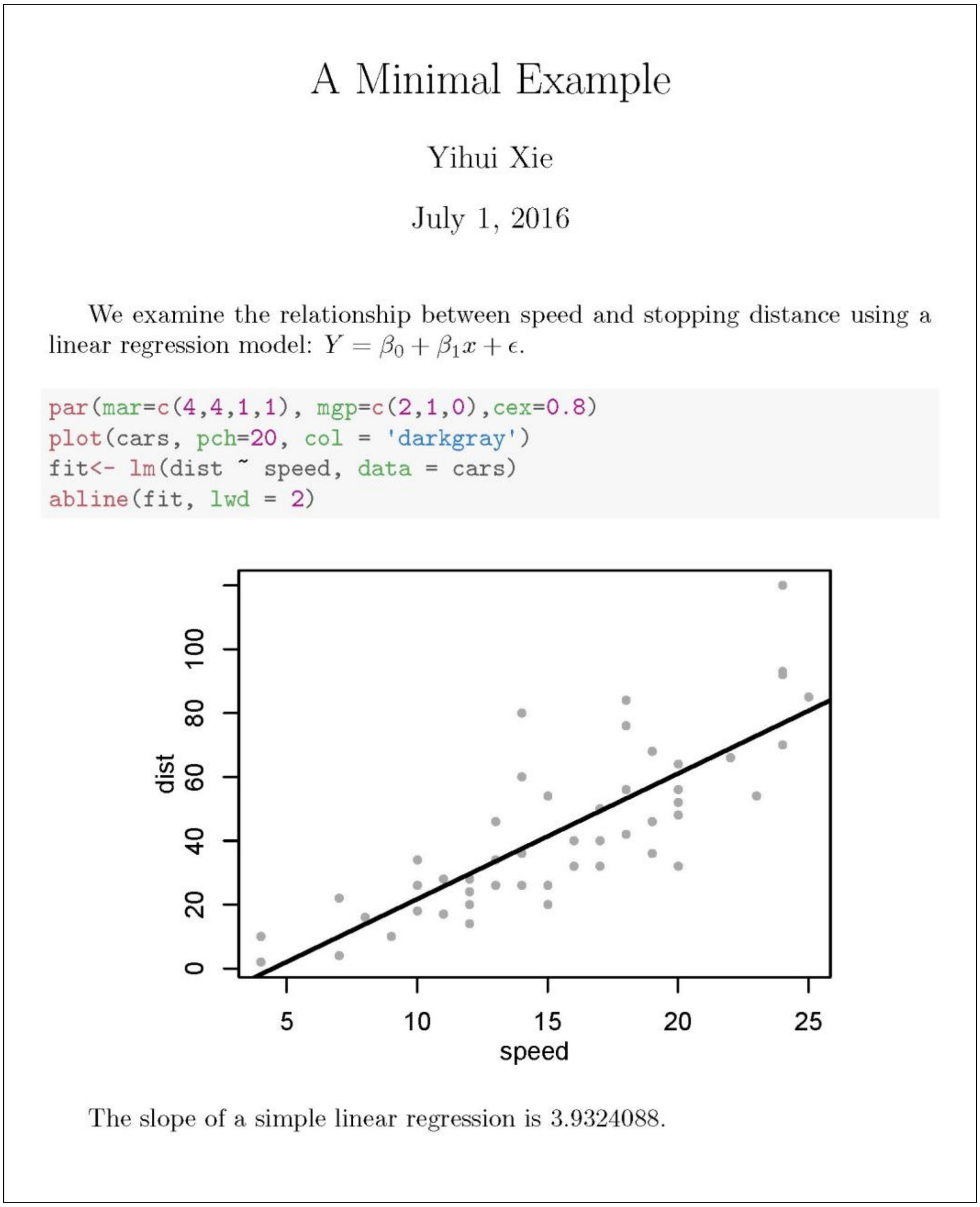
R code embedded in LaTeX code is saved as a .Rnw file, essentially just a text file. Xie (2015) provides a minimal example:

```
\documentclass{article}
\begin{document}
\title{A Minimal Example}
\author{Yihui Xie}
\maketitle

We examine the relationship between speed and stopping distance using a linear regression model:
$Y=\beta_0 + \beta_1 x + \epsilon$.

<<model, fig.width=4, fig.height=3, fig.align='center'>>=
par(mar=c(4,4,1,1), mgp=c(2,1,0),cex=0.8)
plot(cars, pch=20, col = 'darkgray')
fit<- lm(dist ~ speed, data = cars)
abline(fit, lwd = 2)
@

The slope of a simple linear regression is
\Sexpr{coef(fit)[2]}.
\end{document}
```

In R, running the ‘knit' function on the .Rnw file produces a .tex file, which can be compiled in the usual way to produce a PDF. If using RStudio, this process can be reduced to a single button click (see Section 5.3). For the minimal example, this produces the document shown in Figure 2. The first few lines of code will be familiar to LaTeX users. The R code lies between the ’<<>>=' and ‘@' markers - these define where the R code starts and ends. Sections of code are referred to as ‘chunks'. In this example, the R code sets some graphical parameters, plots the ‘cars' data (supplied with R), fits a linear model, then adds the fitted regression line to the plot. Options for the chunk are given between the ‘<<' and ‘>>=' markers. Here they give it a name: ‘model' so it can be referred to elsewhere, and set the size and alignment of figures produced by the chunk. The penultimate line embeds inline code, giving the regression coefficient in this case.

5.2. Updating Documents and Further Options

If I want to change the dataset that has been used, or if the dataset has been updated to a more recent version, I simply make the appropriate change to the ‘data' argument within the R code chunk and re-run knitr to create the updated output document. If I was using other statistical and word processing packages, for example SPSS and Microsoft Word, I would likely need to manually re-do the analysis and change all of the figures and results. Similarly if I want to make a change to a figure, I don't have to re-make the figure, save or copy it, and insert it into the document. I just make the change in the R code chunk and re-run knitr. When the data or analyses are changed, it is easy for errors to occur in the document as a consequence of some results not being updated to the latest version. Here, this issue does not arise as everything is updated automatically.



A Minimal Example

Yihui Xie

July 1, 2016

We examine the relationship between speed and stopping distance using a linear regression model: $Y = \beta_0 + \beta_1 x + \epsilon$.

```
par(mar=c(4,4,1,1), mgp=c(2,1,0),cex=0.8)
plot(cars, pch=20, col = 'darkgray')
fit<- lm(dist ~ speed, data = cars)
abline(fit, lwd = 2)
```


The slope of a simple linear regression is 3.9324088.

Figure 2. Output of minimal example

You can use the chunk options to either show or hide code and output for different audiences or uses. For example, if you are making introduction to statistics worksheets for non-statistics students, you might not want to include R code. If you are including exercises for students, you might not want to include the output and instead have them answer some questions. To hide the R code, use the option ‘echo=FALSE', to hide the output, use the option ‘eval=FALSE'. It is also possible to set these options globally so they apply to the whole document, therefore by changing one option, you can have two different versions of the document - one with code/output and one without. This could be useful if you are creating worksheets and want to create a version with the solutions included.

While it is the focus of this article, a PDF is not the only possible output from knitr. For users wanting to create webpage content, Markdown can be used instead of LaTeX, with R code embedded in the same way. Instead of saving as a .Rnw file, code is saved as a .Rmd file and the output is Markdown which can be used to produce HTML. See Xie (2015) for further details.

5.3. Setting Up RStudio for knitr

The first step is to install R, the additional R package, knitr and RStudio (RStudio is optional but this section assumes you will be using it). Then you need to set up RStudio for knitr. Go to tools-> Global options, click the 'sweave' tab on the left (Figure 3) and change 'weave Rnw files using' to knitr. These steps only need to be completed once when first setting up your system. Having done this, when you have a .Rnw file open in RStudio, you should see a 'compile PDF' button on the toolbar. Clicking this will run the knit function on the file, then run pdflatex on the resultant .tex file, giving you the final PDF output. It will also run bibtex if you are using it for references within your document. So knitting the .Rnw file, compiling the .tex file and running bibtex is all achieved with one button click!

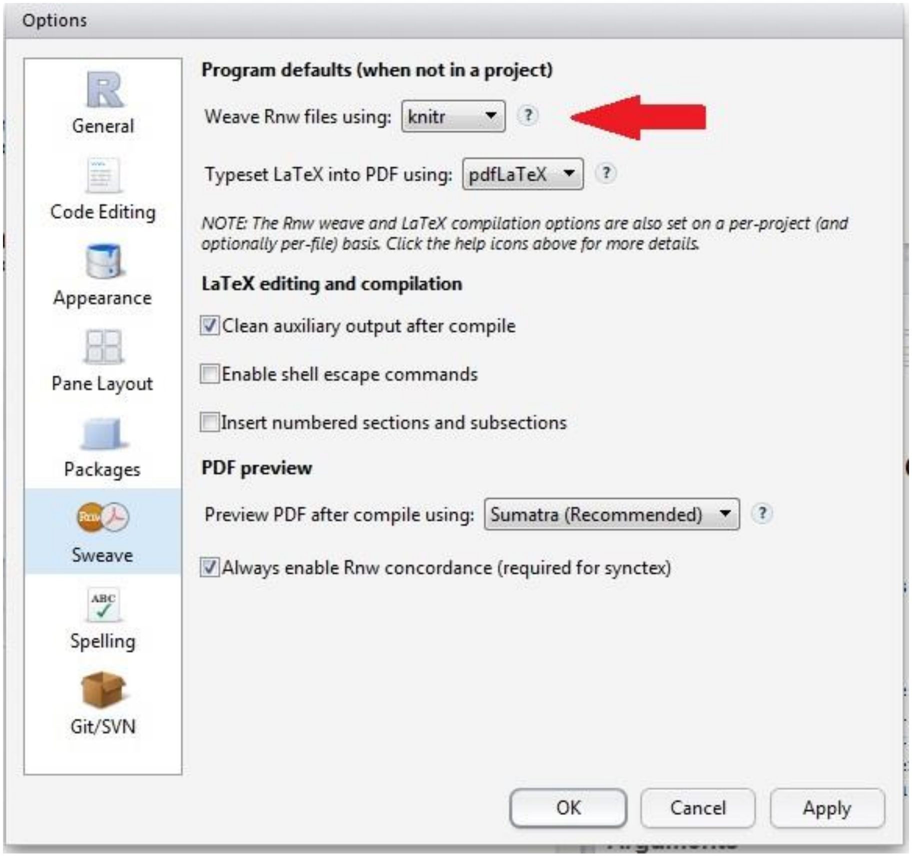


Figure 3. Setting RStudio options for knitr

6. More Examples

Let's look at some more examples using the cars dataset from R. This dataset contains the speed and stopping distances of 50 cars, recorded in the 1920s. The following code will produce a histogram. We just include the code within a code chunk and the histogram will automatically be produced and included in the final document.

```
hist(cars$speed, main="Speed of cars in the 1920s", xlab="Speed (mph)")
```

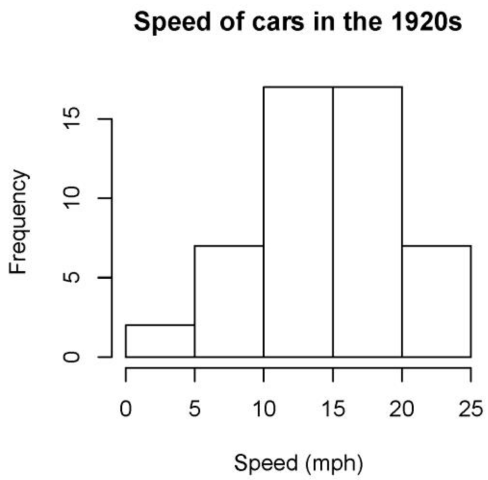


Figure 4. Automatically generated histogram

Some descriptive statistics:

```
colMeans(cars)
 ##   speed   dist
 ##   15.40   42.98
apply(cars,2,sd)
 ##   speed         dist
 ##   5.287644      25.769377
```

Again, R code is included in a code chunk and the output is produced automatically, preceded by '##'. For cars in the 1920s, the average speed was 15.4, with a standard deviation of 5.29 and the average stopping distance was 42.98, with a standard deviation of 25.77. Here, as in Xie's minimal example, we use the command Sexpr{} to embed inline code: rather than writing out the means and standard deviations, we include the code for it, so these numbers would automatically update if the data were changed. Another clear advantage is that this reduces the risk of typing errors: we simply indicate where we want the numbers to be and the software inserts them for us.

Many functions are available either in the core R packages, or in additional packages. However, sometimes it is helpful or necessary to be able to write your own. As always when using knitr, we can include the code within a code chunk and the output will be automatically generated and included in the document. For example, this very simple function will square whatever it is given as input.

```
simple<-function(x){  #creates a function named 'simple' which takes x as input
  y<-x^2                #everything inside the curly braces is what the function does
  return(y)             #the function will return the value of y
}
simple(2)               #calls the function, giving a value of 2 for x
## [1] 4
y                       #note that R does not know what y is outside of the function
## Error in eval(expr, envir, enclos): object 'y' not found
```

This is a very simple function and it was not really necessary to create a function: we could simply have written:

```
2^2
## [1] 4
```

But we can extend this function to include more lines of code, more complicated code, anything we want.

```
bigger<-function(x){
  y<-x^2
  z<-y+2
  w<-z^2
  return(c(x,y,z,w))      #note the use of c to return a vector
}

bigger(2)
## [1] 2 4 6 36
```

Additional packages can be installed using the install.packages function, for example, for the advanced graphics package ‘ggplot2':

```
install.packages("ggplot2")
```

Packages only need to be installed once per machine, but need to be loaded into the session each time you use it, using the ‘library' function:

```
library(ggplot2)
```

Along with the use of some other packages, ggplot2 can be used to make some appealing graphics, including a world map. As with the previous examples, including the following code in a code chunk will automatically generate the plot.

```
library(ggmap)

cities <- c("London ", "Sydney", "Paris", "Washington", "Moscow")
ll.cities <- geocode(cities) #find the longitude and latitude
cities.x <- ll.cities$lon
cities.y <- ll.cities$lat

#Using ggplot, plot the world map
mp <- NULL
mapWorld <- borders("world", colour="gray30", fill="lightblue")
mp <- ggplot() + mapWorld
#Add city locations as blue dots
mp <- mp+ geom_point(aes(x=cities.x, y=cities.y) ,color="blue", size=1)
mp
```

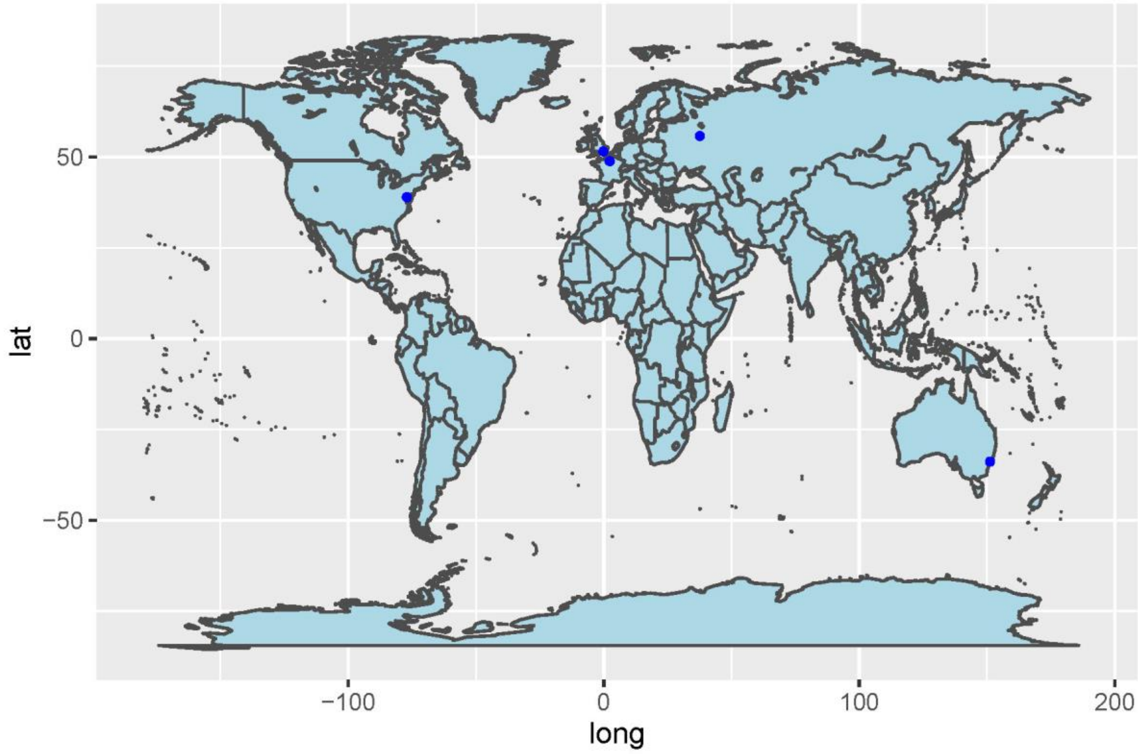


Figure 5. Automatically generated map

7. Further Reading and Information

For further information about knitr, examples and details of available options, see Yihui Xie's book (Xie, 2015) and website (http://yihui.name/knitr/). For examples of knitr reports created by users, see http://rpubs.com.

8. References

Knuth, D.E., 1984. Literate programming. *The Computer Journal*, *27*(2), pp.97-111.

RStudio, 2016. RStudio. www.rstudio.com.

R Core Team, 2015. R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria. Available at: https://www.R-project.org/ .

Xie, Y., 2015. *Dynamic Documents with R and knitr*. Boca Raton: CRC Press.

Xie, Y., 2016. knitr: A General-Purpose Package for Dynamic Report Generation in R. R package version 1.15.1.

OPINION

Football scores, the Poisson distribution and 30 years of final year projects in Mathematics, Statistics and Operational Research

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Abstract

The development of the Poisson match as a model used in the prediction of the outcome of football matches is described. In this context, many interesting modelling projects arise that are suitable for undergraduate, final year students. In a narrative that discusses the author's engagement with this model and other related models, the paper presents a number of these projects, their attractions and their pitfalls, and poses a number of questions that are suitable for investigation. The answers to some of these questions would be worthy of the attention of the administrators of their respective sports.

Keywords: Poisson distribution; sport; competitive balance; tournament design.

1. Background

This paper is a personal story as well as a description of some mathematics. I first met the "Poisson match" in 1983 as a Mathematics undergraduate at Sheffield University. On a project-based module, students read and summarised papers on sports statistics, among which was the now seminal work of Mike Maher on modelling football scores (Maher, 1982). This was the start of more than thirty years of interaction with this model. A few years later, in a final year project that I set as a young lecturer in the early 1990s, two final year mathematics students forecast football match scores using this model in order to play the *Football Pools*. The 'Pools' was a forerunner to the National Lottery. Millionaires who had correctly forecast eight score draws were front page celebrities. In the simplest stake in this game, a player selected eight matches from among all the matches to be played in the English Football League on a given weekend, earning 3 points if the result of a selected match was a score-draw, 2 for a no-score-draw, 1½ for an away win, and 1 point for a home win. Those who scored 24 points shared the jackpot. So, the object was to forecast match scores, and then make the selection on the basis of these forecasts.

Selecting 8 matches at random achieved 14 points on average, and using Maher's model we could improve this to 17, reducing the expected waiting time for a jackpout from 10^7 plays to 10^6 approximately. There are three problems. Firstly, football match scores are inherently difficult to forecast because games are often finely balanced, and arguably more so than any other game. This is one reason why the game is so popular (Forrest and Simmons, 2002; Buraimo and Simmons, 2015), and I will return to this point later. The second problem is shortcomings in the model itself, and indeed Maher's model has been developed and refined by many. The third problem is sparcity of data, so that parameters are not sufficiently well estimated.

To explain this last point, we must describe the Maher model in a little detail. Doing so also suggests many open questions that are suitable for student projects. It is immediately apparent that the score of a team A in a match M is a random variable taking values 0,1,2,.., and therefore the Poisson distribution is a candidate model for this score. To use this model, the mean score (the parameter of the Poisson distribution) must be estimated. Now obviously, when A plays team B at venue C at time *t* say, this is a *random trial* that has never been observed before. Even if A played B at venue C in the previous season, conditions would be different, the teams would be different, a different referee, weather, etc. So what data should we use? Ignoring the issues just raised and assuming all matches AvB at C are statistically identical (and independent), we could use the scores from all the matches AvB at C in the recent seasons. However, this would still give only a small number of datapoints, bearing in mind that the further one goes back in time the more dubious are the assumptions, not least because the abilities or strengths of A and B evolve over time. In fact if one proceeds in this naïve way, for the English Premier League (EPL) now with twenty teams, this approach implies 380 parameters to be estimated. Further, one season's data is itself 380 matches. Maher's trick was to propose that each team possesses an attacking strength (or tendency to score goals) and a defensive weakness (or tendency to concede goals), and the mean score of A when A plays B is the product of A's attacking strength, α_1, and B's defensive weakness, β_2, and a home advantage parameter, δ say, which does not depend on the venue. Thus, if X_1 is the random variable that denotes the score of A when A plays B at home, and X_2 is the score of B in this match, then $\mathrm{E}(X_1)=\alpha_1\beta_2\delta$ (and $\mathrm{E}(X_2)=\alpha_2\beta_1$), and X_1 and X_2 have independent Poisson distributions. With this model, there are two parameters per team, plus the home advantage, making 41 in total for the EPL (actually only 40 because strength is relative). Nonetheless, estimation remains a problem because teams' strengths evolve over time—some teams improve, others decline. Owen (2011) and Koopman and Lit (2015) model this strength evolution. McHale and Kharrat (BBC Sport, 2017) take a different approach, using player line-ups to determine the strengths. The independence of scores is questioned by Dixon and Coles (1998), although McHale and Scarf (2007, 2011) found only slight negative dependence. Now I may be digressing slightly here, but it is important to depict the modelling landscape in order to develop ideas for further related projects.

Returning to my own relationship with the Maher model, a decisive moment was the opportunity to develop the statistical model that underlies the EA Sports Player Performance Index, which is described in McHale et al. (2012). This fortunate event rekindled my work in this area, and further student projects and some publications followed, connected with tournament design (Scarf et al., 2009), cricket (Scarf et al., 2011) and my own passion for mountain running and orienteering (Scarf, 2007). My latest work concerns the question of competitive balance in rugby union, which also uses the Poisson distribution as its basis.

So where is this narrative leading and what has it to do with final year projects for undergraduates? Well, my first point is that the sports industry wants analysts (Brady et al., 2017), not least to repeat the 'Moneyball' success (Lewis, 2003). My second is that sport provides interesting projects, for which data are widely available and easily collected. The third is that sport gives students the opportunity for the application of modelling to the real world (Porter and Bartholomew, 2016). The fourth is that while developments of Maher's model are too difficult for undergraduate projects, many simpler, related questions remain, and arise contemporaneously. Indeed, in the next section, in which I describe open questions that would make suitable projects, I begin with the recent decision of FIFA to extend the soccer World Cup finals. The open questions are presented as something of a list, organised around broad headings, with some discussion of how they might be tackled, and what technical issues may arise and how they can be avoided. The paper finishes with the rugby union question. I hope this list will provide useful inspiration for teachers of undergraduates. I am not aware of a work that has set out to classify projects in this way.

2. Open questions and projects

2.1. Tournament design

FIFA has announced that the World Cup finals in 2026 will comprise 48 teams rather than 32. The immediate question is what effect this will have on the tournament. Obviously it will be larger (more matches), but what will be the effect on its competitiveness? A rather simple argument is that there will be none because the strength of the teams ranked, during the qualification stage, between 33rd and 48th in the world are only very marginally weaker than those ranked say 17th to 32nd. Actually, one can speculate that a greater proportion of matches in the tournament as a whole will be more balanced. What will be the effect on the tournament outcome? The simplest way to study this is to calculate the probability that the best team wins using simulation. A final year project might take the form: *'Use simulation to study the effect on the probability that the best team wins of increasing the tournament size from 32 to 48'.* Another interesting question is: *'How does the number of unimportant matches vary with the size of the tournament?'* A match is deemed unimportant if its outcome has no effect on the tournament outcome, e.g. in a group stage a match between teams who are already eliminated. To tackle these questions, the Maher model can be used to simulate individual matches, using either team strengths that are estimated by the investigator using data collected by the investigator (a more difficult variation of the project) or team strengths that have been reported by others in published work and that are assumed typical of the tournament studied (a less difficult variation of the project). In the less difficult project, the investigator might assume that team strengths of additional teams are the same as those ranked 17 to 32, for example. Or existing ranking lists might be used to modify strengths slightly. In both variations, it is necessary to code: assignment of teams to groups and matches, match outcomes between known teams; progression of teams to knockout stages; match outcomes decided following a tie at full-time (a 'coin-toss' simplest); and repetitions of the complete tournament.

Many variations on this project are then possible: '*Investigate different forms of tournament for the soccer World Cup finals*'. Here, different combinations of group sizes, number of group rounds, and number of knockout rounds can be investigated. Other projects can look at variation in rules for seeding, '*Investigate the effect of different seeding procedures on the probability that the best team wins*'. Interesting questions relate to the UEFA Champions League: its pre-tournament qualification rounds; and the appropriate number of qualifying teams from each national league. These are timely questions because sports administrators continue to fiddle with tournament design.

Reproduction of the results of Maher (1982) would make a starting point for a more demanding project that attempts to investigate developments of the Maher model and its estimation. Other sports tournaments, in e.g. cricket, rugby union, tennis, can be investigated using the model of Bradley and Terry (1952). Within this theme softer projects are possible, for example: '*Carry out a comparative study of tournament design across European football leagues*'; '*Discuss the relative merits of the different designs that have been used in the cricket World Cup'.*

2.2. Rule Changes

Studies of tournament design changes are similar in scope to studies of changes to scoring systems. Thus one might set a project to: '*Study the effect of changes in the scoring system in badminton*', focusing on the scoring rules before and after 2006, when scoring only on serve changed to scoring regardless of serve, and games from first to 15 to first to 21. It turns out that this question has been studied by (Percy, 2009). However, a large number of variations on this theme in any one of a number of sports are possible. The investigator might even propose a favourite sport to investigate in what-if analysis. Such studies require a simple model for winning a point, on serve and against serve. A *Bernoulli trial* with a different win probability in each case will do. Tennis, with its archaic scoring system, could be investigated: '*Is serve dominance the determining factor in game length*?' This study could contrast tennis, where games are short (to 4) and many (>12), with badminton, where games are long (to 21) and few (3).

2.3. Score distributions and dependency

In football, scores are almost independent. '*In basketball, what is the nature of score dependency?*' Possession changes and high scoring rates suggest strong dependency, but this may not be true. One could '*Classify team ball sports by score dependency',* by collecting scores in high profile tournaments for each sport, calculating correlations between scores, and then attempting to relate these to the nature of the sports and perhaps even their popularity. In sports with many different means of scoring points (e.g. variations of football), one has to determine whether to focus on points or numbers of scoring events. One might '*Investigate dependency between scoring types in sports with multiple scoring modes*'. For example, in rugby union, is the number of tries correlated with the number of penalties? The "Poissoness" of scores might be investigated: '*Investigate in sport S the nature of the distribution of scores*'.

2.4. Competitive balance

The final questions that I consider are presented in a little more detail, and investigate the relationship between outcome uncertainty and scoring rate. They originate from three questions that may themselves be posed in projects: '*What are the essential characteristics of a popular sport?*'; '*Why is football (soccer) so popular worldwide*?'; and '*Is rugby union becoming increasingly uncompetitive?'.* The first two questions here are rather broad. Consequently, they offer possibilities for more or less technical solutions. A softer study might carry out a survey using a questionnaire. A more technical project might relate competitiveness to measures that are surrogates for popularity. In the latter, one requires a measure of competitiveness. Many have been proposed in the sports economics literature (e.g. Utt and Fort, 2002). Here the terms competitiveness, competitive balance, and uncertainty of outcome are used interchangeably.

It is my own conjecture that rugby union is becoming less competitive. If this is true, it is natural to ask '*Why?'* Stepping away from a real sport for a moment, let us construct a mathematical sport in which scores follow independent Poisson distributions, and investigate the relation between competitive balance and scoring rate. Let us call such a game a "pure Poisson match" and investigate the question: '*In a pure Poisson match, what is the relationship between team strengths, scoring rates and uncertainty of outcome?'*

To develop this idea a little further, we need some preliminaries. Let Y be a random variable. Then for any $t \neq 0$, $\Pr(Y \geq 0) = \Pr(e^{tY} \geq 1) \leq E(e^{tY})$ by Markov's inequality (since e^{tY} is a non-negative random variable). Now let $Y = X_1 - X_2$ where X_1 and X_2 are independent. Then

$$\Pr(X_1 - X_2 \geq 0) \leq E(e^{t(X_1 - X_2)}) = E(e^{tX_1})E(e^{-tX_2}) . \quad (1)$$

Now suppose $X_1 \sim \mathrm{Po}(\lambda_1)$ and $X_2 \sim \mathrm{Po}(\lambda_2)$ independent, so that (X_1, X_2) is a pure Poisson match. The moment generating function of X_i is given by $E(e^{sX_i}) = \exp\{\lambda_i(e^s - 1)\}$. Therefore from (1) we obtain

$$\Pr(X_1 - X_2 \geq 0) \leq \exp\{\lambda_1(e^t - 1) + \lambda_2(e^{-t} - 1)\} .$$

Now setting $t = \log(\sqrt{\lambda_2 / \lambda_1})$, we obtain

$$\Pr(X_1 - X_2 \geq 0) \leq \exp\{\lambda_1(\sqrt{\lambda_2 / \lambda_1} - 1) + \lambda_2(\sqrt{\lambda_1 / \lambda_2} - 1)\} = \exp\{-(\sqrt{\lambda_1} - \sqrt{\lambda_2})^2\}.$$

Now let $\lambda_1 = \lambda$ and $\lambda_2 = \varepsilon\lambda$ for some $\varepsilon > 1$, so that team 2 is slightly stronger than team 1. Then we have that $\Pr(X_1 \geq X_2) \leq \exp\{-\lambda(1-\sqrt{\varepsilon})^2\} \to 0$ as $\lambda \to \infty$ for fixed ε. Therefore $\Pr(X_1 < X_2) \to 1$ as $\lambda \to \infty$ for fixed ε. Therefore, in a pure Poisson match, no matter how close are the strengths of the two teams, in the limit (for a very large scoring rate) the stronger team will always win and the match is perfectly competitively unbalanced. Further, if $\lambda_1 = \lambda_2 = \lambda$ then $\Pr(X_1 = X_2) \to 0$ as $\lambda \to \infty$ (proof omitted) and $\Pr(X_1 < X_2) = \Pr(X_1 > X_2) \to \frac{1}{2}$ as $\lambda \to \infty$ (by symmetry). A technical project might ask for proofs of these results.

The above deals with the asymptotic behaviour of $\Pr(X_1 \geq X_2)$. For the exact calculation of $\Pr(X_1 \geq X_2)$ for $X_1 \sim \mathrm{Po}(\lambda)$ and $X_2 \sim \mathrm{Po}(\varepsilon\lambda)$ we can use

$$\Pr(X_1 < X_2) = \sum_{y=1}^{\infty} \sum_{x=0}^{y-1} \{e^{-\lambda}\lambda^x / x!\}\{e^{-\varepsilon\lambda}(\varepsilon\lambda)^y / y!\}, \qquad (2)$$

and

$$\Pr(X_1 = X_2) = \sum_{x=0}^{\infty} \{e^{-\lambda}\lambda^x / x!\}\{e^{-\varepsilon\lambda}(\varepsilon\lambda)^x / x!\} = \sum_{x=0}^{\infty} e^{-\lambda(1+\varepsilon)}\lambda^{2x}\varepsilon^x / (x!)^2 . \qquad (3)$$

A less mathematical, more empirical project might use these exact formulae to illustrate the asymptotic results graphically. The probabilities (2) and (3) could also be evaluated by simulation.

Now the question of dependence of scores in particular sports is a pertinent one. An '*Investigation of the relationship between scoring rates and outcome uncertainty in a double Poisson match*' would make an interesting, empirical study. In the double Poisson match (Karlis and Nzoutfras, 2003) scores are not independent. One might speculate that scores in rugby union show some dependence, but in spite of this as scoring rates increase outcomes become less certain. Finally, the project to '*Investigate the evolution of scoring rates in rugby union over time*' can shed some light on the state of this sport and what if anything its administrators should do about it.

3. Conclusion

Sport offers many opportunities for projects. This is because data are relatively abundant and easy to obtain, models are relatively intuitive, and the context is often evident to the investigator. There is also a sport for everyone, and arguably a mathematical project exists within every sport. This paper has considered a number of projects that are suitable for final year undergraduates. These projects are unified within the notion of a Poisson match. The projects are by no means trivial, and some pose questions of which administrators of the respective sports should take note. One wonders indeed if administrators use modelling at all to consider proposals for change. Important decisions should be based on evidence and such evidence should be scrutinised by good modellers. Thus as the business of sport grows, there is a growing need for trained modellers to work in sport, and projects on sport are a stepping stone to employment.

4. References

BBC Sport, 2017. Premier League: Chelsea to win by seven points? Arsenal outside top four? Available at: http://www.bbc.co.uk/sport/football/38524102. [Accessed 12 January 2017].

Bradley, R.A. and Terry, M.E., 1952. The rank analysis of incomplete block designs 1. The method of paired comparisons. *Biometrika,* 39, pp.324-345.

Brady, C., Forde, M. and Chadwick, C., 2017. Why your company needs data translators. *MIT Sloan Management Review*, Winter 2017 issue.

Buraimo, B. and Simmons, R., 2015. Uncertainty of outcome or star quality? Television audience demand for English premier league football. *International Journal of the Economics of Business,* 22, pp. 449-469.

Dixon, M. and Coles, S., 1997. Modelling association football scores and inefficiencies in the football betting market. *Applied Statistics,* 46, pp. 265–280.

Forrest, D. and Simmons, R., 2002. Outcome uncertainty and attendance demand in sport: The case of English soccer. *Journal of the Royal Statistical Society: Series D,* 51, pp.229–241.

Karlis, D. and Ntzoufras, I., 2003. Analysis of sports data using bivariate Poisson models. *Journal of the Royal Statistical Society: Series D,* 52, pp.381–393.

Koopman, S. and Lit, R., 2015. A dynamic bivariate Poisson model for analysing and forecasting match results in the English Premier League. *Journal of the Royal Statistical Society: Series A,* 178, pp.167–186.

Maher, M., 1982. Modelling association football scores. *Statistica Neerlandica,* 36, pp.109–118.

Lewis, M., 2003. *Moneyball: The Art of Winning an Unfair Game.* New York: Norton.

McHale, I. and Scarf, P., 2007. Modelling soccer matches using bivariate discrete distributions with general dependence structure. *Statistica Neerlandica,* 61, pp.432-445.

McHale, I. and Scarf, P., 2011. Modelling the dependence of goals scored by opposing teams in international soccer matches. *Statistical Modelling,* 11, pp.219–236.

McHale, I., Scarf, P. and Folker, D., 2012. On the development of a soccer player performance rating system for the English Premier League. *Interfaces,* 42, pp.339-351.

Owen, A., 2011. Dynamic Bayesian forecasting models of football match outcomes with estimation of the evolution variance parameter. *IMA Journal of Management Mathematics,* 22, pp.99–113.

Percy, D., 2009. A mathematical analysis of badminton scoring systems. *Journal of the Operational Research Society,* 60, pp.63-71.

Porter, R. and Bartholomew, H., 2016. When will I ever use that? Giving students opportunity to see the direct application of modelling techniques in the real world. *MSOR Connections,* 14, pp.45-49.

Scarf, P., 2007. Route choice in mountain navigation, Naismith's rule and the equivalence of distance and climb. *Journal of Sports Sciences,* 25, pp.719-726.

Scarf, P., Mat Yusof, M. and Bilbao, M., 2009. A numerical study of designs for sporting contests. *European Journal of Operational Research,* 198, pp.190-198.

Scarf, P., Shi, X. and Akhtar, S., 2011. The distribution of runs scored and batting strategy in test cricket. *Journal of the Royal Statistical Society, Series A,* 174, pp.471-497.

Utt, J. and Fort, R., 2002. Pitfalls to measuring competitive balance with Gini coefficients. *Journal of Sports Economics,* 3, pp.367-373.

CASE STUDY

Experience sharing: Mathematical Contest in Modelling (MCM)

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Abstract

In January 2016, Coventry University's **sigma** Mathematics Support Centre (MSC) funded three students for MCM, a multi-day mathematics competition held annually in the USA. This is organised by the Consortium for Mathematics and Its Applications (COMAP) and sponsored by the Mathematical Association of America (MAA), the Society of Industrial and Applied Mathematics (SIAM) and the Institute for Operations Research and the Management Sciences (INFORMS). In this article the team leader and advisor reflect on their experience.

Keywords: mathematical contest in modelling, real-world problems, interdisciplinary research.

1. Introduction

The MCM is a four-day international mathematics competition for high school students and undergraduates, organised by the Consortium for Mathematics and Its Applications (COMAP), a US-based charity with the aim to improve mathematics education for students of all ages. This involves real-world mathematical modelling where research, analytics and applied intelligence reign along with less-quantifiable factors like timing and luck. It challenges teams of students to clarify, analyse, and propose solutions to open-ended problems. The contest attracts a diverse range of students and faculty advisers from over 900 institutions around the world (see MCM homepage). The MCM is designed to encourage effective discussion supporting informed modelling decisions, improved student problem solving, and to promote technical writing. Students participate as team members rather than as individuals, creating an environment for peer learning and skills development. The MCM has been increasingly popular with rapidly rising numbers of teams participating (Table 1).

Table 1. Numbers of teams participated for the last six years

Year	2011	2012	2013	2014	2015	2016
Number of teams participated	2,775	3,697	5,636	6,755	7,636	12,446

In January 2016, we entered the first Coventry University team, whose student members included Ji Wang (2nd year Mathematics and Statistics), Kaiyuan Lin (3rd Finance exchange student), ChingYi Ng (1st year Mathematics student), along with Dr Aiping Xu, the Manager of Coventry University's sigma Mathematics Support Centre (MSC), as an advisor. In this short article, we reflect upon our experience.

2. The Contest

2.1. Before the contest was open

As a team we went through many past questions and analysed our strengths and weaknesses. The problems tend to be open-ended, and are drawn from all fields of science, business and public policy. Preparation for the contest was in excess of five months (approximately five hours per week per team-mate) and entailed extensive literature review on Mathematical Modelling, Machine Learning, Simulation and Programming, anling with additional topics in mathematics and statistics. This intensive research enabled the team to greatly expand their knowledge of these topics. Coventry University's MSC was of great assistance during this period, having tutors covering a wide range of areas of expertise.

2.2. Our Project

For the contest we were offered a list of six problems (http://www.comap.com/undergraduate/contests/mcm/contests/2016/problems/), consisting of three on mathematical modelling and three on interdisciplinary modelling. After careful consideration, we chose an interdisciplinary modelling problem: 'Are we heading towards a thirsty planet?' (2016 ICM Problem E), which was of interest to the whole team. Moreover, it fell on the areas of data analysis and optimisation, which we have confidence in.

We were asked to address the following six tasks:

1. To develop a model to evaluate a country's ability to produce clean water, which should take into account the dynamic nature of factors that affect both supply and demand.
2. To choose a country or region from the UN water scarcity map (Figure 1) and analyse its water condition using the model built in Task 1.

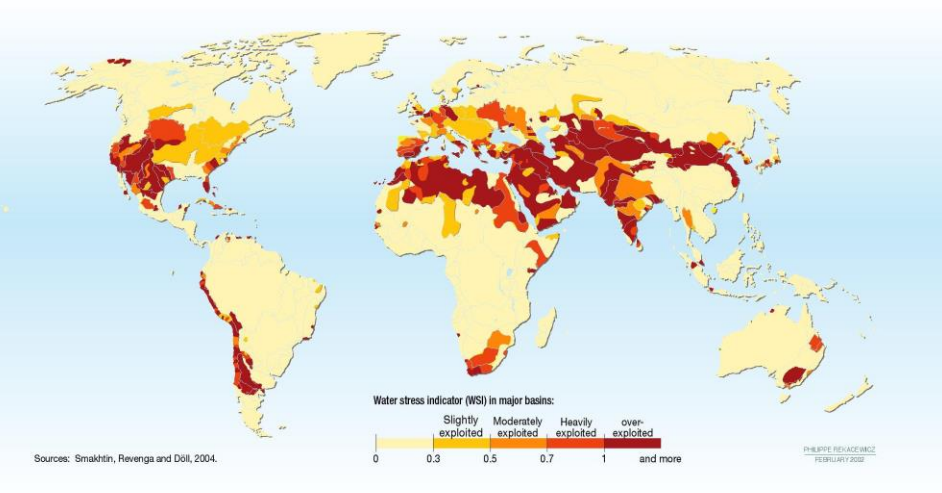

Figure 1. UN water scarcity map

3. To predict what the water situation will be like in 15 years.
4. To design an intervention plan to improve the water situation in the chosen area based on our clean water production ability model built in Task 1.
5. To evaluate the water condition after imposing our intervention plan on the chosen area.
6. To discuss the advantages and disadvantages of our models.

For other users to practise, we recommend starting from the Analytic Hierarchy Process, which provides good theory for the evaluation of problems like these (Saaty 2008).

2.3. Data collection

Clean water production ability can be affected by society, economics and environmental conditions. To evaluate their effects, we carried out some basic analysis, for instance, consideration of the Gross Domestic Product (GDP) as a factor for the economics condition. All economic data was sourced from the World Bank (the World Bank homepage). Regarding water condition data, we checked all the websites of national water resource departments of potential target countries.

2.4. Solution

We started with a Comprehensive Evaluation Model (Figure 2), which enabled us to assess Clean Water Production Ability (CWPA) in light of Water Availability Indicators (WAI). All of the acronyms in Figure 2 are defined in Table 2.

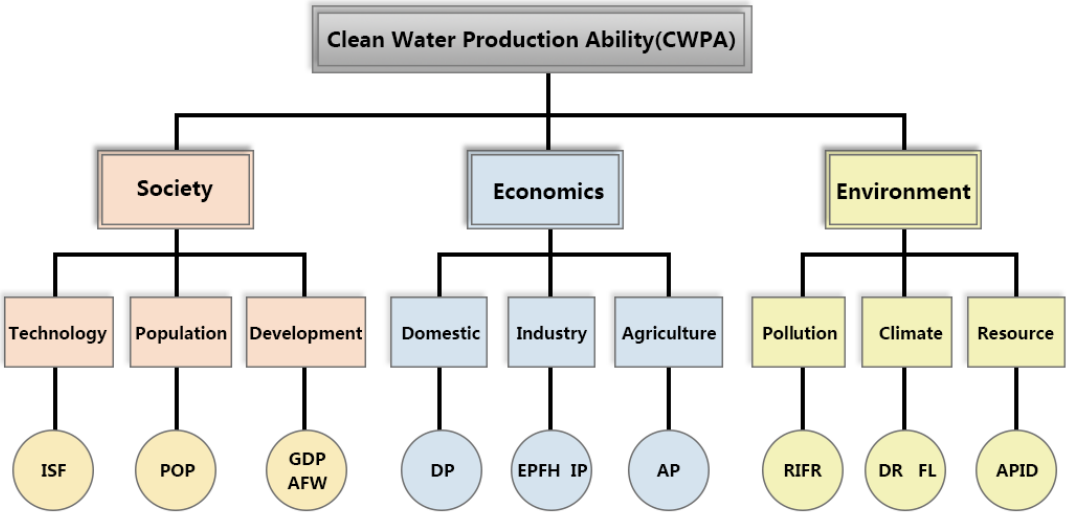

Figure 2. Comprehensive Evaluation Model

Table 2. Definition of acronyms

Abbreviation	Meaning
CWPA	Clean Water Production Ability
DP	Annual Freshwater Withdrawals, domestic (% of total freshwater withdrawal)
AP	Annual Freshwater Withdrawals, Agriculture (% of total freshwater withdrawal)
ISF	Improved Sanitation Facilities (% of population with access)
FL	Flood Occurrence (index)
DR	Drought Severity (index)
EPFH	Electricity Production from Hydroelectric Sources (% of total)
POP	Population in the area
GDP/POP	GDP per capita (dollar)
LA/POP	Land Area per capita (square kilo meter)
AFW/POP	Annual Freshwater Withdrawals per capita (10^3 cube meter)
RIFR/POP	Renewable Internal Freshwater Resources per capita (billion cubic meters)
APID/POP	Average Precipitation in Depth per capita (mm per year)
IP	Annual Freshwater Withdrawals, Industry (% of total freshwater withdrawal)

The model is based on Multiple Linear Regression, which has been proven to be accurate and robust through sensitivity testing. Germany was then chosen as an object for observation because its water resources are moderately exploited in the UN water scarcity map (Figure 1). We have analysed the change in indicators over time, which is essential for determining the future trend for other models. The current situation is interpreted with the model representation and empirical evidence.

Our forecast models included Logistic Regression and AutoRegressive Integrated Moving Average (ARIMA) models for the estimation of indicators. With the Comprehensive Evaluation Model, we could simulate CWPA changes in 15 years without any intervention. A comparison between present and future situations is then made to evaluate the impact of changes including environmental effects on citizens.

We then set up our Best Development Plan and an alternative Minimum Requirement Plan for Germany using Non-Linear Programming. Our Best Development Plan is to figure out the optimal combinations of inductors that lead to the optimal CWPA. The alternative way is to work out the minimum changes in inductors to achieve a given level of water availability. The feasibility of all indicators is explained in this part.

We finished with an Influence Model, which was adopted to estimate the effect of the intervention plan on surrounding areas. Based on Graph Theory, Geopolitics and Game Theory, our Influence Model could measure the systematic influence between countries quantitatively (Figure 3). The weights in the graphical model in Figuare 3 are measured by the ratio of the GDPs between the adjoined countries, which describes the influence of power. The arrows demonstrate the influence direction between countries. For example, Germany is the country that influences all its adjoining countries. The Czech Republic is most influenced by Germany. All the influences shown in the graphical model are negative because we have assumed that all the countries play a zero-sum game.

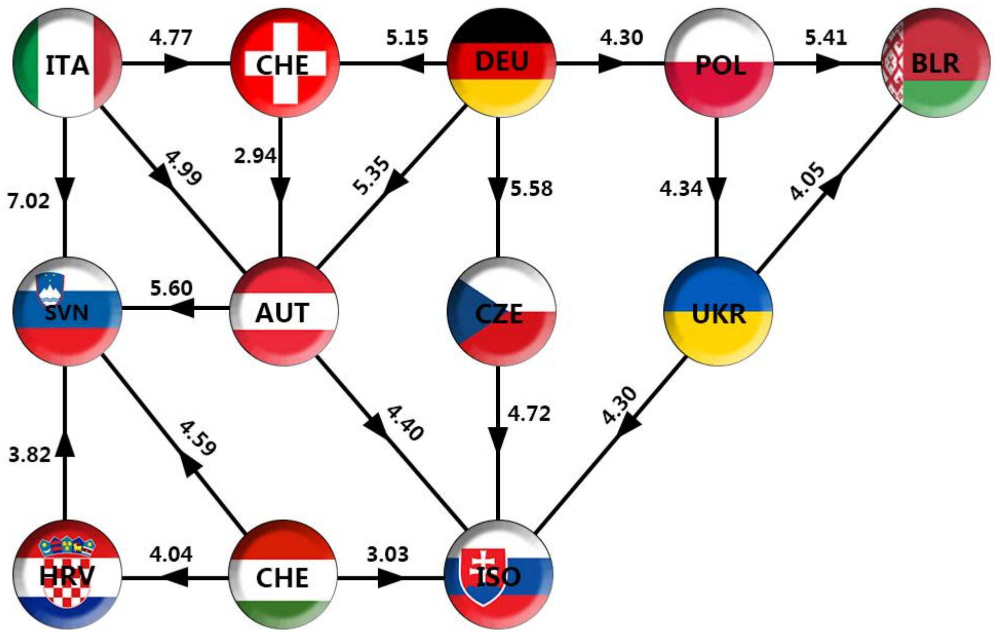

Figure 3. Directed graph with weights in adjusted log-form

2.5. *Strength and Weakness of our models*

Strengths:

- Different levels of a minimum development plan can be worked out based on different targets in the future, which can be taken into consideration for macro-policy making.
- Most of the proposals are proved to be doable in both the Best Development Plan and the Minimum Development Plan based on analysis.
- The Influence Model measures the systematic influence between countries. The basics are derived from geopolitical concepts.

Weaknesses:

- Restrictions are difficult to set accurately. The range is sometimes vague, depending on historic data.
- The assumptions on the Influence Model are too theoretical.
- Lack of data makes some estimation less accurate in the forecast models.

3. Our achievement

There were 12,446 teams from 900 institutions around the world participating in 2016. The awards and percentages were: Outstanding Winners (0.2%), Finalist Winners (0.3%), Meritorious Winners (12.3%), Honourable Mentions (39.3%), and Successful Participants (46.9%). We are delighted to be a Meritorious Winner (Figure 4).

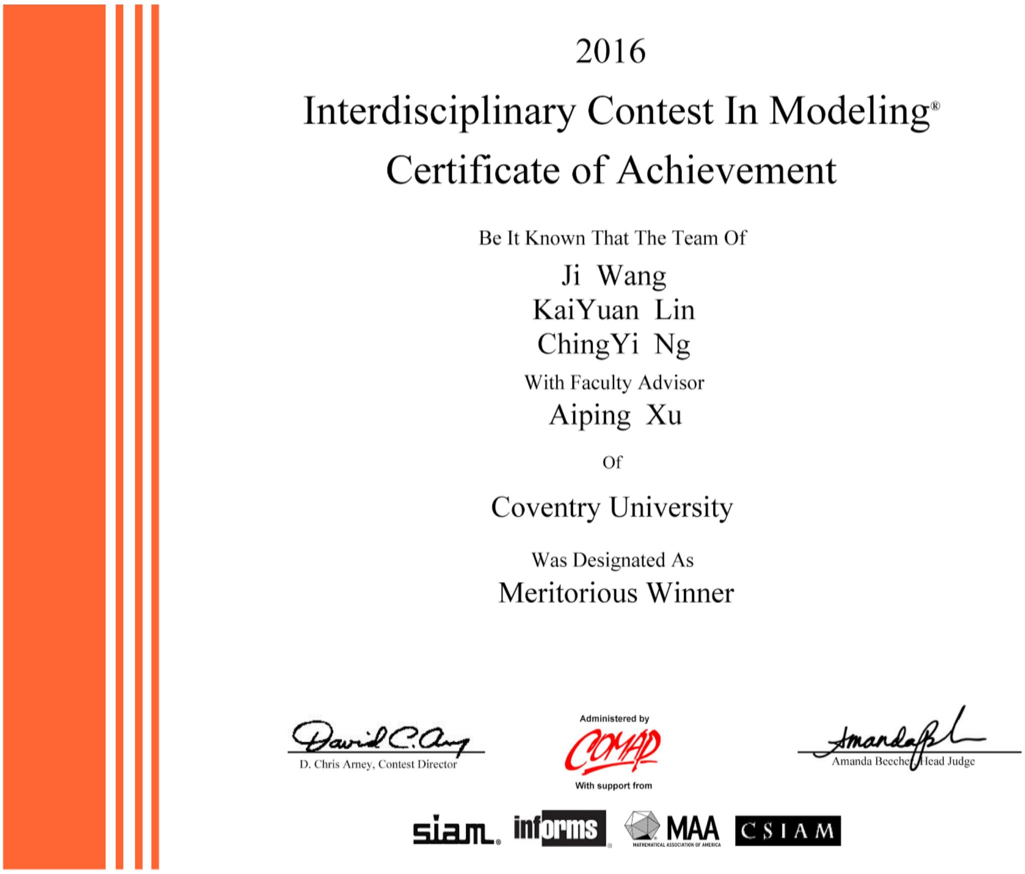

Figure 4. Winner Certificate

4. Reflection

On reflection, working out how to apply mathematics and statistics to real-world problems was the contest's most challenging aspect. In order to complete the tasks within the given time frame, we had to comprehend the problems and quickly develop suitable approaches, as well as to read through relevant literature so as to better understand the background and methodologies of related fields. Finding a complete and accurate water condition dataset for the model was more difficult than anticipated. However, we have thoroughly enjoyed the contest. It greatly improved our understanding of mathematics and statistics, especially with respect to solving real-world problems. It required knowledge outside of Mathematics and Statistics (e.g. Ecology and Hydrology for our problem), the development of which was challenging but satisfying. Additionally, our ability to communicate and cooperate effectively within a group whilst under pressure was hugely improved by taking part in the contest.

In hindsight, we would change the following aspects:

- To choose better teammates. It would be beneficial if all the teammates contribute evenly so careful selection of the team is really important.
- To determine which question to focus on at an earlier stage, instead of considering all of them, which turned out to be very time-consuming. We had only 96 hours to research and submit our solutions in the form of a research paper thus time was a big issue.
- To read more widely before the contest. As our problem is interdisciplinary, better models could be produced if we had gained more knowledge of other subjects.

This international contest provided us with opportunities and challenges. We strongly recommend wider participation, where many different skills can be learnt and consolidated.

5. Acknowledgements

All team members are extremely grateful for the support (including financial) that they obtained from the **sigma** Mathematics Support Centre.

6. References

MCM homepage. Available at: http://www.comap.com/undergraduate/contests/ [Accessed 12 October 2016].

2016 ICM Problem E: Are we heading towards a thirsty planet? https://www.comap.com/undergraduate/contests/mcm/contests/2016/problems/2016_ICM_Problem_E.pdf [Accessed 12 October 2016].

Satty, T. L., 2008. Decision making with the analytic hierarchy process, *International Journal of Services Sciences*, 1(1), pp.83-98

The World Bank homepage. Available at: http://www.worldbank.org/ [Accessed 12 October 2016].